Π^0_1-LEM

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Notations

HA^ω_e : (extensional)Heyting arithmetic in all types.

HA can be viewed as a subsystem of (a definatorial extension of) HA^ω_e.

Types are denoted by $\sigma, \rho, \tau, \cdots$. $\rho \to \tau$ is also expressed as $\tau\rho$.

$AC : \forall x^\rho \exists y^\tau A(x,y) \to \exists y^{\rho\to\tau} \forall x^\rho A(x, yx)$ (A arbitraty)

$(E^2) : \exists \Phi^2 \forall f^1 (\Phi(f) \overset{0}{=} 0 \Leftrightarrow \forall x^0 (f(x) \overset{0}{=} 0)))$

For all (ordinary) primitive recursive predicates $R(x) \in \mathcal{L}(HA)$ (with possibly additional number parameters) we have the schema:

$\Sigma^0_1 - LEM : \exists x R(x) \vee \forall x \neg R(x)$

$\Pi^0_1 - LEM : \forall x R(X) \vee \neg \forall x R(x)$

$MP_{pr} : \neg\neg \exists x R(x) \Rightarrow \exists x R(x)$

Note In the following, the parameters in $R(x)$ may be omitted.

Proposition 1 $HA^\omega_e + AC + (E^2) \nvdash MP_{pr}$

The proof of Proposition 1 will be presented later. We will first state two of its corollaries.

Corollary 1 $HA^\omega_e + AC + \Pi^0_1 - LEM \nvdash \Sigma^0_1 - LEM$

Corollary 2 $HA + \Pi^0_1 - LEM \nvdash \Sigma^0_1 - LEM$

Here is a diagram indicating these relations.

We will show how to prove the arrows in the diagram.

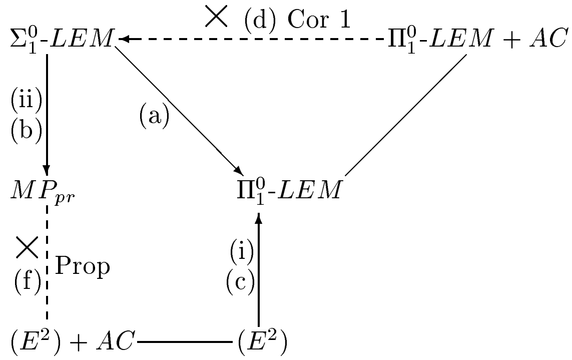
(a) $\Sigma^0_1 - LEM \to \Pi^0_1 - LEM$

$P0$:

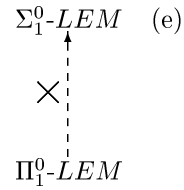
$$\cfrac{\cfrac{\cfrac{\neg R(x), R(x) \to}{\neg R(x), \forall x R(x) \to}}{\exists x \neg R(x) \to \neg \forall x R(x)}}{\exists x \neg R(x) \to \forall x R(x) \vee \neg \forall x R(x)}$$

Put $\Pi^0_1(R) \equiv \forall x R(x) \vee \neg \forall R(x)$.

In HA_e^ω ::



Cor 2 In HA ::



$$\dfrac{P0 \qquad \dfrac{\forall x R(x) \to \forall x R(x)}{\forall x R(x) \to \Pi^0_1(R)}}{\underbrace{\exists x \neg R(x) \vee \forall x \neg\neg R(x)}_{\Sigma^0_1(\neg R)} \to \Pi^0_1(R)}$$

(b) $\Sigma^0_1 - LEM \to MP_{pr}$

$P1$:

$$\dfrac{\dfrac{\exists x R(x) \to \exists x R(x)}{\neg\neg\exists x R(x), \exists x R(x) \to \exists x R(x)}}{\exists x R(x) \to \underbrace{\neg\neg\exists x R(x) \Rightarrow \exists x R(x)}_{MP(R)}}$$

$P2$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\neg R(x), R(x) \to}{\forall x \neg R(x), R(x) \to}}{\forall x \neg R(x), \exists x R(x) \to}}{\forall x \neg R(x), \neg\neg\exists x R(x) \to}}{\forall x \neg R(x), \neg\neg\exists x R(x) \to \exists x R(x)}}{\forall x \neg R(x) \to \neg\neg\exists x R(x) \Rightarrow \exists x R(x)}$$

$$\dfrac{P1 \quad P2}{\underbrace{\exists x R(x) \vee \forall x \neg R(x)}_{\Sigma^0_1(R)} \to MP(R)}$$

(c) $(E^2) \to \Pi^0_1$-LEM

$f_R x = 0 \leftrightarrow R(x)$ (axiom in HA_e^ω)

We will temporarily omit the R in f_R. $E(\Phi, f)$:

$\Phi(f) = 0 \Leftrightarrow \forall x(fx = 0)(\Leftrightarrow \forall x R(x))$

$P1$:

$$\frac{E(\Phi, f), \Phi(f) = 0 \to \forall x(fx = 0) \to \forall x R(x)}{E(\Phi, f), \Phi(f) = 0 \to \Pi^0_1(R)} \; (\vee I : \neg\forall R(x))$$

$P2$

$$\frac{E(\Phi, f), \neg\Phi(f) = 0 \to \neg\forall x(fx = 0) \to \neg\forall x R(x)}{E(\Phi, f), \neg\Phi(f) = 0 \to \Pi^0_1(R)} \; (\vee I : \forall x R(x))$$

$$\cfrac{\overset{(decidable)}{\overset{\downarrow}{\to}} \quad \Phi(f) = 0 \vee \neg\Phi(f) = 0 \qquad \cfrac{P1 \quad P2}{E(\Phi, f), \Phi(f) = 0 \vee \neg\Phi(f) = 0 \to \Pi^0_1(R)}}{\cfrac{E(\Phi, f) \to \Pi^0_1(R)}{\underbrace{\exists\Phi\forall f E(\Phi, f)}_{(E^2)} \to \Pi^0_1(R)}}$$

Using these facts, we will prove the corollaries.

Proof of Corollary 1, i.e. (d):

Assume first the following facts. (See the diagram.)

Lemma 1 (b), i.e. (ii) $HA^\omega_e + \Sigma^0_1 - LEM \to MP_{pr}$
(c), i.e. (i) $HA^\omega_e + (E^2) \to \Pi^0_1 - LEM$
(f) $HA^\omega_e + AC + (E^2) \nrightarrow MP_{pr}$ (Proposition 1)

Now suppose

$$HA^\omega_e + AC + \Pi^0_1 - LEM \to \Sigma^0_1 - LEM \qquad (g)$$

By (c) and (g) we obtain
(h) $HA^\omega_e + (E^2) + AC \to \Sigma^0_1 - LEM$ (cut out $\Pi^0_1 - LEM$)
By (b) and (h)

$$HA^\omega_e + (E^2) + AC \to MP_{pr}$$

(cut out $\Sigma^0_1 - LEM$)
This contradicts (f) (Proposition 1).
Therefore, (g) does not hold, qed.

Proof of Corollary 2, i.e. (e):
Suppose, contrary to (e),

$$HA + \Pi^0_1 - LEM \to \Sigma^0_1 - LEM$$

Then, since we can regard $HA \subset HA^\omega_e$,

$$HA^\omega_e + \Pi^0_1 - LEM + AC \to \Sigma^0_1 - LEM$$

contradicting Corollary 1.

In order to prove Proposition 1, i.e. (f), we give the definition of *s-majorization* in [1], p.1221.

Definition 1 (cf. [1]) $x^\rho \leq_\rho y^\rho$ and $z^\rho s - maj_\rho x^\rho$ will be defined as follows.

$$\begin{cases} x_1 \leq_0 x_2 :\equiv (x_1 \leq x_2) \\ x_1 \leq_{\tau\rho} x_2 :\equiv \forall y^\rho (x_1 y \leq_\tau x_2 y) \end{cases}$$
$$x_i : \rho \to \tau (or \tau\rho)$$

$$\begin{cases} x^* s - maj_0 x :\equiv x \leq_0 x^* \\ x^* s - maj_{\tau\rho} x :\equiv \forall y^{*\rho}, y^\rho \end{cases}$$
$$(y^* s - maj_\rho y \to x^* y^* s - maj_\tau x^* y \wedge x^* y^* s - maj_\tau xy)$$

($s - maj_\rho$ is a variation of W.A.Howard's maj)

Examples

$$0 \to 0 : \quad x_1 \leq_1 x_2 :\equiv \forall y^0 (x_1 y \leq_0 x_2 y)$$
$$x^* \equiv \lambda z^0.(z^0 + 1)$$
$$x \equiv \lambda z^0.z^0$$
$$(\underbrace{y^* s - maj_0 y}_{y^* \geq y} \to \underbrace{x^* y^* s - maj_0 x^* y, xy}_{y^*+1 \geq y+1, y^*+1 \geq y})$$

Counter-example

$$x^* = \lambda z^0 (3 \dot{-} z)$$
$$x = \lambda z (2 \dot{-} z)$$
$$y^* = 3, y = 2$$
$$x^* y^* = 0, x^* y = 1$$
$$\therefore x^* y^* \not\geq x^* y$$
$$y^* = 3, y = 1$$
$$x^* y^* = 0 \not\geq xy = 1$$

For $\rho = 0 \to 0$, $x^* s - maj_\rho x$ means that $x^*$is greater than x as values, and non-decreasing. (${x^* y^*}_0 \geq x y^*$ follows from the case $y^* = y$)

Find a Φ^2, whose existence is guaranteed by (E^2), and then put $\Phi^*(f) = \min(\Phi^2(f), 1)$.

$$\begin{cases} \Phi^2(f) = 0 \quad \text{implies} \quad \Phi^*(f) = 0 \\ \Phi^2(f) \geq 1 \quad \text{implies} \quad \Phi^*(f) = 1 \end{cases}$$

We next give the definition of *monotone mr*.

Definition 2 (Monotone mr) $\mathbf{t}^*$ is said to satisfy the *monotone modified realizability interpretation* of $A(\mathbf{a})$ (written as $\mathbf{t}^* mr_{mon} A(\mathbf{a})$) if

$$\exists \mathbf{x}(\mathbf{t}^* maj \mathbf{x} \wedge \forall \mathbf{a}(\mathbf{x}\mathbf{a} mr A(\mathbf{a})))$$

(maj stands for Howard's majorization in [2].)

Concerning the monotone mr-interpretation, we have the

Theorem 1 (Realization of the theorems) If $HA_e^\omega + AC + (E^2) \vdash A(\mathbf{a})$, then there are closed terms $\mathbf{t}^*$ of HA_e^ω such that

$$HA_e^\omega + (E^2) \vdash \mathbf{t}^* mr_{mon} A(\mathbf{a})$$

Proof By induction on the complexity of the proof-figure leading to $A(\mathbf{a})$. The realizations of the axioms of HA_e^ω and of the ac are well-known. The inference rules preserve the monotonte mr-realizability.

(Note that it is important that there is a formal proof. So, the x in $\exists x$ needs not be explicitly obtained.)

For (E^2), we can take the following. $t^* = 1^2 := \lambda f^1.1^0$ satisfies the mr_{mon}-interpretion of (E^2):

$t^* mr_{mon}(E^2)$ (under the assumption (E^2)) means

$$1^2 mr_{mon}(E^2) \leftrightarrow \exists x(1^2 maj x \wedge \forall a(xa mr(E^2)))$$

As x, we can take the Φ^* above.

(Since (E^2) is closed, we may assume that there is no occurrence of a.)

Back to Proof of (f) Suppose, contrary to (f),

$$HA_e^\omega + AC + (E^2) \vdash MP_{pr}$$

Then, taking $T(x, x, y)$ as $R(x, y)$,

$$HA_e^\omega + AC + (E^2) \vdash \neg\neg\exists y T(x, x, y) \Rightarrow \exists y T(x, x, y)$$

$A(x)$ is $\neg\neg\exists y T(x, x, y) \Rightarrow \exists y T(x, x, y)$.

By Theorem above,we can extract a closed term t^1 of HA_e^ω such that

$$HA_e^\omega + (E^2) \vdash t^1 mr_{mon} A(x)$$

that is,

$$\exists u(t^1 maj\ u \wedge \forall x(ux\ mr A(x))),$$

or

$$\exists u(\forall x, t^1 x \geq ux \forall x(\neg\neg\exists z T(x, x, z) \Rightarrow T(x, x, ux)))$$

From this, we obtain

$$\forall x \exists y \leq t^1 x(\neg\neg\exists z T(x, x, z) \Rightarrow T(x, x, y)) \qquad \text{(j)}$$

We will show the following.

$$HA_e^\omega + (E^2) \vdash \forall x(\exists y T(x,x,y) \Leftrightarrow \exists y \leq tx T(x,x,y)) \qquad \text{(k)}$$

$$\exists y \leq f(x) T(x,x,y) \Rightarrow \exists y T(x,x,y) \qquad \text{(l)}$$

and

$$\exists y T(x,x,y) \Rightarrow \neg\neg\exists z T(x,x,z) \qquad \text{(m)}$$

are trivially proved. In general,

$$\exists y \leq f(x)(B(x) \Rightarrow T(x,x,y) \rightarrow B(x) \Rightarrow \exists y \leq f(x) T(x,x,y) \qquad \text{(n)}$$

Proof of (n)

$$\cfrac{\cfrac{\cfrac{\cfrac{B(x), B(x) \Rightarrow T(x,x,y) \rightarrow T(x,x,y) \quad y \leq f(x) \rightarrow y \leq f(x)}{y \leq f(x), B(x), B(x) \Rightarrow T(x,x,y) \rightarrow y \leq f(x) \wedge T(x,x,y)}}{y \leq f(x), B(x), B(x) \Rightarrow T(x,x,y) \rightarrow \exists y \leq f(x) T(x,x,y)}}{y \leq f(x), B(x) \Rightarrow T(x,x,y) \rightarrow B(x) \Rightarrow \exists y \leq f(x) T(x,x,y)}}{\exists y \leq f(x)(B(x) \Rightarrow T(x,x,y)) \rightarrow B(x) \Rightarrow \exists y \leq f(x) T(x,x,y)}$$

With $f \equiv t^1$,

$$\begin{cases} \exists y T(x,x,y) \rightarrow \neg\neg\exists z T(x,x,z) & \text{(m)} \\ \exists y \leq f(x)(\neg\neg\exists z T(x,x,z) \Rightarrow T(x,x,y)) & \text{(j)} \end{cases}$$

$$\rightsquigarrow \exists y \leq f(x)(\exists y T(x,x,y) \Rightarrow T(x,x,y))$$

$$\rightsquigarrow \exists y T(x,x,y) \Rightarrow \exists y \leq f(x) T(x,x,y) \qquad \text{(p)}$$

The last step follows from (m) with $B(x) \equiv \exists y T(x,x,y)$.

(l) and (p) prove (k). But this would imply the decidability of the halting problem by a function in Gödel's T. For:

$$HA_e^\omega + (E^2) \simeq \text{Gödel's T}$$

So, if $\forall x(\cdots)$ is provable in $HA_e^\omega + (E^2)$, then $\exists y T(x,x,y)$ is decided by $y \leq f(x)$, yielding a contradiction.

Note to the proof

The main fact we showed in the proof above was that, if

$$HA_e^\omega + AC + (E^2) \vdash \forall x^0(A(x) \Rightarrow \exists y^0 R(x,y))$$

then there is a closed term t^1 such that

$$HA_e^\omega + (E^2) \vdash \forall x \exists y \leq t^1 x(A(x) \Rightarrow R(x,y))$$

if A is an $\exists$-free formula.

This is a very special case of Th3.10 (and Cor.3.11 in p.1228) in [1], where $(HA_e^\omega = E - PA_i^\omega)$

References

[1] U. Kohlenbach, *Relative Constructivity*, JSL 63(1998).

[2] A. S. Troelstra, *Mathematical Investigation of Intuitionistic Arithmetic and Analysis*, LNM 344(1973), Springer-Verlag.