An arithmetical hierarchy of the law of excluded middle and related principles

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1 Introduction

The topic of this paper is Relative Constructivism. We are concerned with classifying non-constructive principles from the constructive viewpoint. We compare, up to provability in Intuitionistic Arithmetic, sub-classical principles like Markov's Principle, (a function-free version of) Weak König's Lemma, Post's Theorem, Excluded Middle for simply Existential and simply Universal statements, and many others. Our motivations are rooted in the experience of one of the authors with an extended program extraction and of another author with bound extraction from classical proofs.

Motivated by the approximation interpretation in [1], one of the authors found that many non-constructive proofs could be better understood as constructions of more general kind, as "limit constructions" or "learning processes" [5, 7]. This allowed also some kind of "execution", even for results for which no construction exists (in the more traditional sense of the word). For instance, Hilbert finite basis theorem cannot produce a generator for a given ideal. However, it may be interpreted as a process learning, by trial-and-error, the generator of the ideal. If we may interpret also non-constructive theorems as more general kind of construction, the first question arising is: how do we classify this new set of constructions? When a mathematical principle is, from an intuitionistic viewpoint, but a particular case of another one? We will prove, for instance, that Weak König's Lemma and Markov's principle are mutually independent, and that their conjunction is weaker than Excluded Middle, even just over simply Universal statements. And the different strength of principles represent different models of constructions. Proofs by weaker principles carry more information to understand constructions hidden in proofs. The executions of learning processes extracted from proofs are expected to be useful for "debugging of proofs" [6, 5]. Then it is better to prove theorems by principles as weak as possible. The classification of the principles in the paper are expected useful to determine by which principles we should prove the target theorem.

An independent motivation for studying, from a constructive viewpoint, the relations between semi-classical principles was extraction of recursive bounds. Consider any classical proof of some arithmetical statement $\forall x.\exists y.A[x,y]$. A bound for this statement is a map f such that, for all $x \in N$, $A[x,y]$ for some $y \leq f(x)$. It is well-known that, in general, we cannot extract out of the proof a recursive bound f, unless A is purely existential. However, if a proof of a result of classical mathematics only uses certain restricted classical principles such as comprehension for existential-free formulas or (weak) König's lemma in an overall intuitionistic context, then the extraction of a recursive bound f for an arbitrary A is possible (see [11]). If only weak König's lemma is used (but no non-recursive comprehension) the intuitionistic context may even be extended by the Markov's principle. This was another motivation for studying the relation, in an intuitionistic context, between König's lemma and Markov's principle, and between their conjunction and Excluded Middle for Universal statements.

2 Semi-classical Principles

In this section, we introduce the *semi-classical principles* and discuss on their motivations and formulations. We investigate classical logical principles such as the law of excluded middle **restricted to arithmetical fragments**, e.g. the law of excluded middle relativized to Σ^0_n formulas. The restriction makes classical principles somehow more "effective" and provides finer understanding of the "computational" nature of classical principles. We will give motivation for the semi-classical principles relating them to mathematical theorems and concurrent processing.

We work with a standard formulation of Heyting arithmetic HA, i.e., the equational axioms, defining equations for all primitive recursive functions, $S(x) = 0 \supset \bot$, the induction axiom scheme, and the intuitionistic first order logic. Thus, $\mathsf{PA} = \mathsf{HA}+$the law of exculuded middle, where PA stands for Peano arithmetic.

2.1 Definitions of the principles

First we recall that Σ^0_k-formula and Π^0_k-formula are defined as follows. :

- Σ^0_k-formula is $Q_1x_1.\cdots.Q_kx_kP(x_1,\dots,x_k)$.
- Π^0_k-formula is $\overline{Q_1}x_1.\cdots.\overline{Q_k}x_kP(x_1,\dots,x_k)$.

Here P is a quantifier-free formula. Q_i represents $\exists$ for odd i and $\forall$ for even i. $\overline{\forall}$ and $\overline{\exists}$ are $\exists$ and $\forall$, respectively. For example, a Σ^0_2-formula is of the form $\exists x_1.\forall x_2.P(x_1,x_2)$ and a Π^0_2-formula is of the form $\forall x_1.\exists x_2.P(x_1,x_2)$

Definition 2.1 *(i) A $\Pi^0_n \vee \Pi^0_n$-formula is of the form $C_1 \vee C_2$ (C_1, $C_2 \in \Pi^0_n$). (ii) A* bounded Σ^0_{n+1}-formula, *or a $B\Sigma^0_{n+1}$-formula, is of the form $\exists x \le y.C$ ($C \in \Pi^0_n$)*

We introduce *semi-classical principles* by relativizing classical logical principles to formula classes. Let Φ, Ψ be sets of formulas of HA, then we define semi-classical principles for Φ as follows:

(Φ-DNE) $\neg\neg P \supset P$ ($P \in \Phi$)

(Φ-LEM) $P \vee \neg P$ ($P \in \Phi$)

(Φ-LLPO) $\neg(P \wedge R) \supset P^\neg \vee R^\neg$ ($P, R \in \Phi$)

($\Delta(\Phi,\Psi)$-LEM) $(P \Leftrightarrow R) \supset P \vee \neg P$ ($P \in \Phi$, $R \in \Psi$)

$P^\neg$ stands for "duals". For example, $(\forall x.\exists y.x = y)^\neg$ is $\exists x.\forall y.\neg x = y$. $P^\neg$ is defined only for prenex normal forms in this paper. Thus, we assume P and R in LLPO scheme are prenex normal forms.

DNE stands for "double negation elimination", and **LEM** for "law of the excluded middle". **LLPO** stands for "lesser limited principles of omniscience". There are several different but provably equivalent formulations of LLPO schemes for higher degrees. The current one, which is due to Michael Toftdal, is temporarily chosen.

We will consider semi-classical principles only for mathematically or computationally meaningful Φ and Ψ. The principles considered in this paper are Σ^0_n-**DNE**, Π^0_n-**DNE**, $B\Sigma^0_n$-**DNE**, Σ^0_n-**LLPO**, $(\Pi^0_n \vee \Pi^0_n)$-**DNE**, and $\Delta(\Sigma^0_n,\Pi^0_n)$-**LEM**). The principle $\Delta(\Sigma^0_n,\Pi^0_n)$-**LEM** is normally denoted by Δ^0_n-**LEM**.

Note that

(Σ^0_1-DNE) $\neg\neg\exists x.P \supset \exists x.P$ ($P \in \Pi^0_0$)

is the Markov principle for quantifier-free formulas. We also call Σ^0_n-**LEM** as *n-Markov principle*. The principles are often written in the following form:

(Σ^0_n-LEM) $\exists x.P \vee \neg\exists x.P$ ($P \in \Pi^0_{n-1}$)

to make the important logical signs stand out, in this case, the existential quantifier.

Σ^0_1-**LLPO** is called (arithmetical) **LLPO** or **LNOS**, and studied by Bishop school [4]. Σ^0_n-**LLPO** is called n-**LLPO** as well. $B\Sigma^0_2$-**DNE** was considered in [19].

Many principles we considered immediately reduce each other

Fact 2.2 $(\Pi^0_n \vee \Pi^0_n)$-**DNE** $\vdash_{\mathsf{HA}}$ Π^0_n-**DNE** ${}_{\mathsf{HA}}\dashv\vdash_{\mathsf{HA}}$ Σ^0_{n-1}-**DNE**, Σ^0_n-**LLPO** $\vdash_{\mathsf{HA}}$ Σ^0_{n-1}-**LEM**, $(\Pi^0_n \vee \Pi^0_n)$-**DNE** $\vdash_{\mathsf{HA}}$ Σ^0_{n-1}-**LEM**

Since Π^0_n-**DNE** and Σ^0_{n-1}-**DNE** are essentially the same principle, we seldom consider Π^0_n-**DNE** and consider Σ^0_n-**DNE**, instead.

These principles are formulated mainly for the prenex normal forms. Since the prenex normal form result does not hold for the intuitionistic logic, the reader will wonder if the formulation is enough for the intuitionistic logic. In the subsection 2.3, we will show that the prenex normal forms are enough for our aim, since our formulas have prenex normal forms under the semi-classical principles we consider.

2.2 Interpreting the principles

If we restrict ourselves to degree 1 formulas, the semi-classical principles we listed have an interpretation in term of mathematical results (provided we add some form of comprehension or Choice, as we shall see). Thus, the study of their relative strength, with respect to HA configure itself also as an intuitionistic version of Reverse Mathematics. Some results for theorems of analysis in the vein have been given in [16] and other results in analysis, algebra and logic are also known.

The same principles have, as we anticipate in the introduction, an interpretation in term of "limit construction" or learning processes (see [7]). We first list non-constructive semi-classical principle. Then we consider two non-intuitionistic principles which are still "constructive" in some sense: they allow to extract skolem functions from proofs $\forall\exists$-statements. They are Markov's principle and Post's Theorem.

Σ^0_1-LEM $\exists x. P \vee \neg\exists x. P$ is equivalent, as we will prove, to Excluded Middle for degree 1 formulas. From a mathematical viewpoint, if we add a defining principle for functions, it is equivalent to *Yasugi's* "Gauss' staircase principle": there exists a map taking a real number x, and returning the largest integer n below x. From the viewpoint of mathematics based on Learning, instead, Σ^0_1-**LEM** corresponds to the most general form of learning, in which the number of times we are forced to discard a hypothesis has no computable bound.

Π^0_1-LEM $\forall x. P \vee \neg\forall x. P$. This is a restricted version of Excluded Middle for degree 1 formulas. If we add a defining axiom for functions, it allows to define some non-recursive maps with output 0 or 1 (hence having a recursive bound). From the viewpoint of mathematics based on Learning, it corresponds to a more restricted form of learning, in which the number of times we are forced to discard a hypothesis has some computable bound.

Σ^0_1-LLPO $\neg(\exists x.P \wedge \exists x.R) \supset \forall x.\neg P \vee \forall x.\neg R$. This is the most important principle we consider in this paper. It is equivalent to **WKL**, provided we add choice axiom for simply universal formulas (from now on, to be called Π^0_1-$AC^\vee$). Large parts of mathematics can be carried out in weak base systems plus **WKL**: Cauchy-Peano, Hahn-Banach (for separable spaces), Brouwer and Schauder's fixed point theorems, attainment of maximum of a continuous map on $[0, 1]$, and so forth. With respect to all other principle we consider, it has a curious feature: it has a nice non-deterministic interpretation, Lifschitz's Realization Interpretation. The non-determinism comes out of the fact that, if we assume $\neg(\exists x.P \wedge \exists x.R)$, we may deduce either $\forall x.\neg P$, or $\forall x.\neg R$, or both. In this last case, a Realization interpretation of the principle may choose any of the two possibilities. From the viewpoint of mathematics based on Learning, it corresponds to an even more restricted form of learning, in which the number of times we are forced to discard a hypothesis has some computable bound, *and* the we are learning a non-deterministic map having a negatively decidable graph.

Δ^0_1-LEM $(\exists x.P \Leftrightarrow \forall x.R) \supset \exists x.P \vee \neg\exists x.P$. This is equivalent to the Recursive Comprehension Axiom under Choice axiom for Δ^0_1-predicates (from now on, to be called Δ^0_1-AC). It is also equivalent to formalization Post's Theorem of recursion theory: every positively and negatively decidable set of integers is decidable. A convincing learning interpretation for Δ^0_1-**LEM** is still missing.

Σ^0_1-DNE $\neg\neg\, \exists x. P \supset \exists x. P$. As we said, it is called Markov's principle. Fix any partial recursive map f, and call f convergent in x, if it f terminates in a finite number of steps for the input x. Call f divergent in x if it terminates in no finite number of steps for the input x. Then Markov's principle may be expressed as: if a partial recursive map f is not divergent in x, then f is convergent in x. [1] In Bishop's constructive Analysis, the same statement reads as: "if two recursive reals are not equal, then they are apart". Here equal means that for all $n \in N$, $|x - y| < 2^{-n}$, while apart means that for some $n \in N$, $|x - y| \geq 2^{-n}$.

2.3 A Prenex Normal Form Theorem

In this subsection, we justify the formulation of semi-classical principles restricted to prenex formulas. We will define formula classes E_k and U_k, which are generalizations of the Σ^0_k-formulas, Π^0_k-formulas, respectively. They are intuitionistically equivalent to Σ^0_k-formulas, Π^0_k-formulas under appropriate semi-classical principles. In the end, for example, Π^0_n-LEM and its generalization U_n-LEM are logically equivalent under HA. Thus, our seemingly restricted formulation does not loose the generality. U_k and E_k are defined by alternations of sign quantifier occurrences. We define some auxiliary notions for it.

Definition 2.3 *We associate a* sign $+$ *or* $-$ *to each occurrence of quantifiers in a formula A. (i)* $\exists$ *of a positive (negative) subformula* $\exists x.B$ *has the sign* $+$ $(-)$*. (ii)* $\forall$ *of a positive (negative) subformula* $\forall x.B$ *has the sign* $-$ $(+)$*.*

[1] Markov principle states the convergence of an algorithm without giving any explicit recursive bound (say, a polynomial or primitive recursive bound). As we pointed out, such recursive bound exists provided we do not combine Markov's principle with a fragment of Excluded Middle larger than Σ^0_1-**LLPO**.

We count alternations of signs of nested quantifier occurrences. To do so, we consider all sequences of the nested quantifier occurrences starting from the outermost occurrences and ending with the innermost occurrences. For example,

$$(\exists x.\forall y.((\forall z.x = z) \vee (\neg\forall z.\forall u.\forall v.x > z))) \wedge (\forall a.x = a),$$

has three atomic formulas, and so it has three such sequences: "$\exists\forall\forall$" for $(\exists * .\forall * .((\forall z.x = z) \vee *)) \wedge *$, "$\exists\forall\forall\forall\forall$" for $(\exists * .\forall * .(* \vee (\neg\forall * .\forall * .\forall * .x > z))) \wedge *$, and "$\forall$" for $* \wedge (\forall * .x = a)$. We replace the occurrences with their signs and have "$+--$", "$+-+++$", and "$-$", respectively. By replacing adjacent signs of the same kind by a single sign, we have "$+-$", "$+-+$", and "$-$". We call them the sign alternation paths *of the formula.*

Definition 2.4 (U_k, E_k, P_k, and F_k) *The* degree *of a formula is the maximum of the length of its sign alternation paths. The class of degree k formulas is denoted by F_k. A* U_k-formula *is a formula of F_k such that all sign alternation paths of length k start with $-$. An* E_k-formula *is a formula of F_k such that all sign alternation paths with length k start with $+$. A* P_k-formula *is an F_k-formula not belonging to U_k or E_k either.*

Note that F_0, E_0, U_0, and P_0 are the class of quantifier-free formulas. Every formula with quantifier occurrences is classified into exactly one of E_{n+1}, U_{n+1}, and P_{n+1} as in Figure 1. E_n and U_n are generalizations of Σ^0_n and Π^0_n. Note that our definitions of Π^0_n and Σ^0_n are not cumulative. If it is cumulative, $\forall x.x = y$ is a Σ^0_2-formula. But, we normally call it a Π^0_1-formula and do not call it a Σ^0_2-formula. Thus, we make the definitions non-cumulative, and so the definitions of corresponding U_n and E_n are non-cumulative.

A P_k-formula ($k > 0$) is a propositional combination of U_k-formulas and E_k-formulas, but is itself neither a single U_k-formula nor a single E_k-formula. For example, if $P(x)$, $Q(x)$ stand for atomic formulas, then $\forall x.P(x) \vee \exists x.Q(x)$ is a P_1-formula. Any P_n-formula (classically) represents a Δ^0_{n+1}-predicate.

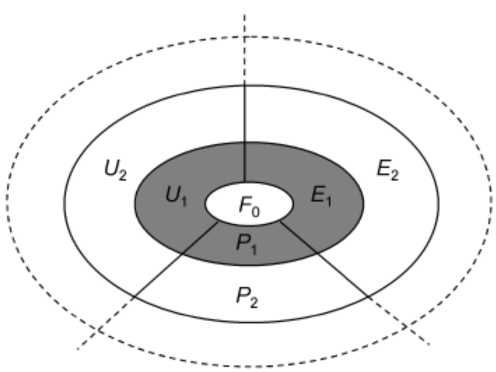

Figure 1. Formula classification

For a supersystem of HA, we say a formula D is *decidable* in T, if $T \vdash D \vee \neg D$.

Fact 2.5 *(i) Any quantifier-free formula is decidable in* HA, *(ii) Boolean combinations of T-decidable formulas are T-decidable.*

Lemma 2.6 *If the variable x does not occur in the formula A, then the following formulas are provable in intuitionistic predicate logic: (i) $(A \vee \neg A) \supset (\forall x(A \vee B) \supset A \vee \forall x.B)$. (ii) $A \vee \forall x.B \supset \forall x(A \vee B)$. (iii) $\exists x(A \circ B) \Leftrightarrow A \circ \exists x.B$, for $\circ = \vee, \wedge$. (iv) $\forall x(A \wedge B) \Leftrightarrow A \wedge \forall x.B$. (v) $D \vee \neg D \supset ((D \supset B) \Leftrightarrow (\neg D \vee B))$, (vi) $(\neg\neg C \supset C) \supset ((B \supset C) \Leftrightarrow (\neg\neg B \supset C))$.*

Theorem 2.7 (Prenex Normal Form Theorem) *(i) If $A \in U_k$, then we can compute a Π^0_k-formula C such that* $(\Pi^0_k \vee \Pi^0_k)$-**DNE** $\vdash_{\mathsf{HA}}$ $A \Leftrightarrow C$, *(ii) If $A \in E_k$, then we can compute a Σ^0_k-formula O such that* Σ^0_k-**DNE** $\vdash_{\mathsf{HA}}$ $A \Leftrightarrow O$.

Proof. The two assertions of the theorem are simultaneously proved by using a double induction on k and the length of A. When $k = 0$, A is atomic and the theorem is trivial. We prove the case for $k > 0$ by assuming the theorem holds below k.

Case 1. A is $A_0 \wedge A_1$: Since Lemma 2.6 allows us to move any quantifiers over the conjunction $\wedge$, a prenex normal form of A is computed in the same way as for classical logic.

Case 2. A is $A_0 \vee A_1$: We prove the assertion (i). By (i) of the induction hypothesis, we have $\forall x_i.O_i \Leftrightarrow A_i$ for some $O_0, O_1 \in \Sigma^0_{k-1}$. ($A_i$ may not be in U_k but in a lower class such as E_{k-1}. Then, we insert some dummy quantifiers to get C_i from the prenex normal form of A_i.) We may assume $x_0 \not\equiv x_1$. The formula

$$\forall x_0.\forall x_1.(O_0 \vee O_1) \supset \neg\neg(\forall x_0.\neg\neg O_0 \vee \forall x_1.\neg\neg O_1)$$

is easily provable in the intuitionistic predicate calculus. Since $(\Pi^0_k \vee \Pi^0_k)$-**DNE** $\vdash_{\mathsf{HA}}$ Σ^0_{k-1}-**DNE**,

$$\forall x_0.\forall x_1.(O_0 \vee O_1) \supset \neg\neg(\forall x_0.O_0 \vee \forall x_1.O_1)$$

is provable in $\mathsf{HA} + (\Pi^0_k \vee \Pi^0_k)$-**DNE**. Applying $(\Pi^0_k \vee \Pi^0_k)$-**DNE** to the conclusion of this implication,

$$\forall x_0.\forall x_1.(O_0 \vee O_1) \supset \forall x_0.O_0 \vee \forall x_1.O_1$$

is provable in $\mathsf{HA} + (\Pi^0_k \vee \Pi^0_k)$-**DNE**. Note that the reverse is provable in the intuitionistic predicate calculus. Since $O_0 \vee O_1$ is an E_{k-1}-formula, it is equivalent to a Σ^0_{k-1}-formula under Σ^0_{k-1}-**DNE** by the induction hypothesis. Obviously, $(\Pi^0_k \vee \Pi^0_k)$-**DNE** $\vdash_{\mathsf{HA}}$

Σ^0_{k-1}-**DNE**. Thus, A is equivalent to a Π^0_k-formula in HA under $(\Pi^0_k \vee \Pi^0_k)$-**DNE**.

We now prove (ii). Similarly as in the proof of (i), we take C_0, $C_1 \in \Pi^0_{k-1}$ such that Σ^0_k-**DNE** $\vdash_{\mathsf{HA}} \exists x.C_i \Leftrightarrow A_i$ $(i = 1, 2)$. Then A is equivalent to $\exists x_0.\exists x_1.(C_0 \vee C_1)$ in HA + Σ^0_k-**DNE**. By Fact 2.2, Σ^0_k-**DNE** implies $(\Pi^0_{k-1} \vee \Pi^0_{k-1})$-**DNE**. Thus, by the induction hypothesis, the U_{k-1}-formula $C_0 \vee C_1$ is equivalent to a Π^0_{k-1}-formula under Σ^0_k-**DNE**. Since repeated existential quantifiers can be combined into a single existential quantifier, this ends the proof of Case 2.

Case 3. $\boldsymbol{A}$ **is** $\boldsymbol{A_0 \supset A_1}$**:** We prove the assertion (i). By the induction hypothesis, we may take $C_0 \in \Pi^0_{k-1}$ and $O_1 \in \Sigma^0_{k-1}$ such that Σ^0_k-**DNE** $\vdash_{\mathsf{HA}} A_0 \Leftrightarrow \exists x_0.C_0$ and $(\Pi^0_k \vee \Pi^0_k)$-**DNE** $\vdash_{\mathsf{HA}} A_1 \Leftrightarrow \forall x_1.O_1$. Since the double negations of **DNE** is intuitionistically provable, $\vdash_{\mathsf{HA}} \neg\neg A_0 \Leftrightarrow \neg\neg\exists x_0.C_0$. Note that $\neg\neg O_1 \supset O_1$ is provable from $(\Pi^0_k \vee \Pi^0_k)$-**DNE**, since $O_1 \in \Sigma^0_{k-1}$. Hence, $(\Pi^0_k \vee \Pi^0_k)$-**DNE** $\vdash_{\mathsf{HA}} \neg\neg\forall x_1.O_1 \supset \forall x_1.O_1$ Thus, by (vi) of Lemma 2.6, $A_0 \supset A_1$ and $\exists x_0.C_0 \supset \forall x_1.O_1$ are equivalent under $(\Pi^0_k \vee \Pi^0_k)$-**DNE**. Since $\exists x_0.C_0 \supset \forall x_1.O_1$ is equivalent to $\forall x_0.\forall x_1.(C_0 \supset O_1)$ in HA and $C_0 \supset O_1 \in E_{k-1}$, the conclusion is easily proved from the induction hypothesis.

We now prove the assertion (ii). Similarly as in (i), there are $O_0 \in \Sigma^0_{k-1}$ and $C_1 \in \Pi^0_{k-1}$ such that $A_0 \supset A_1$ and $\forall x_0.O_0 \supset \exists x_1.C_1$ are equivalent under Σ^0_k-**DNE**. In HA, $\forall x_0.O_0 \supset \exists x_1.C_1$ implies $\neg\neg\exists x_0.\exists x_1.(O_0 \supset C_1)$, the double negations of which can be eliminated by Σ^0_k-**DNE** because of $O_0 \supset C_1 \in U_{k-1}$. Thus, $A_0 \supset A_1$ and $\exists x_0.\exists x_1.(O_0 \supset C_1)$ are equivalent under Σ^0_k-**DNE**. Thus, we can argue as in (i).

Case 4. $\boldsymbol{A}$ **is** $\boldsymbol{\forall x.\, A_0}$ **or** $\boldsymbol{\exists x.\, A_0}$**:** Obvious by the induction hypothesis. □

Corollary 2.8 *(i) Every P_k-formula is decidable in* HA + Σ^0_k-**LEM**. *(ii) For every $H \in P_k$ there exists some $O \in \Sigma^0_{k+1}$ and $C \in \Pi^0_{k+1}$ such that* Σ^0_k-**LEM** $\vdash_{\mathsf{HA}} H \Leftrightarrow O \Leftrightarrow C$.

Corollary 2.9 *(i)* U_k-**LEM** ${}_{\mathsf{HA}}\dashv\vdash_{\mathsf{HA}}$ Π^0_k-**LEM**, *(ii)* E_k-**LEM** ${}_{\mathsf{HA}}\dashv\vdash_{\mathsf{HA}}$ Σ^0_k-**LEM**.

Theorem 2.10 *The prenex normal form theorem is optimal in the following sense: (i) there are $A \in U_k$ and $B \in \Pi^0_k$ such that* $(\Pi^0_k \vee \Pi^0_k)$-**DNE** $\vdash_{\mathsf{HA}} A \Leftrightarrow B$ *and* $A \Leftrightarrow B \vdash_{\mathsf{HA}} (\Pi^0_k \vee \Pi^0_k)$-**DNE**, *(ii) there are $A \in E_k$ and $B \in \Sigma^0_k$ such that* Σ^0_k-**DNE** $\vdash_{\mathsf{HA}} A \Leftrightarrow B$ *and* $A \Leftrightarrow B \vdash_{\mathsf{HA}} \Sigma^0_k$-**DNE**,

Proof. We prove for $k = 1$. The other cases are similarly proved. Let T be Kleene's T-predicate. (i) Set $A \equiv \forall y_0.T(e_0, x, y_0) \vee \forall y_1.T(e_1, x, y_1)$ and $B \equiv \forall y_0.\forall y_1.(T(e_0, x, y_0) \vee T(e_1, x, y_1))$. ($B$ is not really a Π^0_1-formula, but is obviously equivalent to a Π^0_1-formula.) (ii) Set $A \equiv \neg\neg\exists y.T(e, x, y)$ and $B \equiv \exists y.T(e, x, y)$. □

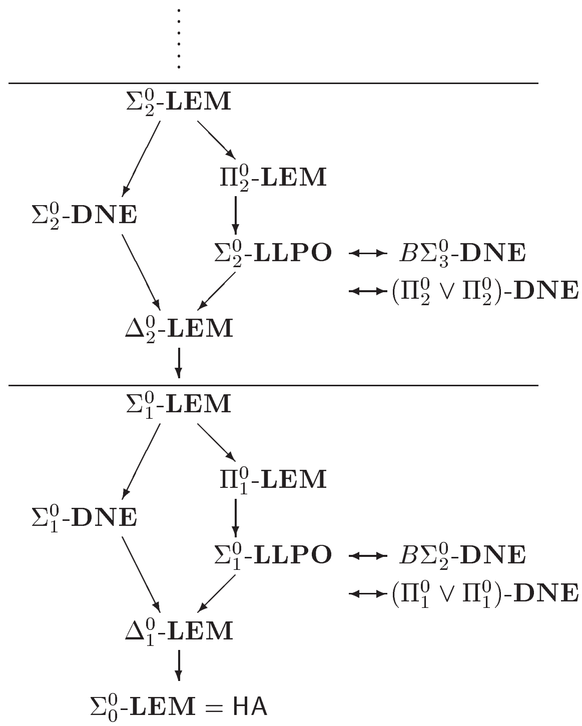

Figure 2. The arithmetical hierarchy

3 Hierarchy of semi-classical principles

In this section, we will determine the provable hierarchy of the semi-classical principles in HA. Semi-classical principles are compared w.r.t. the derivability in HA. This approach resembles the Reverse Mathematics of mathematical logic [15], since we compare logical principles w.r.t. provability in a formal system HA. Strength of set construction principles are considered in the Reverse Mathematics for the aim of the foundation of mathematics. We consider computational strength of logical principles for the practical aims explained in the introduction.

Theorem 3.1 (Main Theorem) *The following hold for Figure 3. (i) For any finite $k > 1$, the arrows (implications) in the figure are derivable in* HA. *(ii)* Σ^0_k-**LEM** *is derivable, if we assume the both of* Π^0_k-**LEM** *and* Σ^0_k-**DNE**. *(iii) When a principle A cannot be reached from another principle B by following the arrows, then A is not derivable from B in* HA. *For example,* Π^0_1-**LEM** $\nvdash_{\mathsf{HA}}$ Σ^0_1-**LEM**, *and* Σ^0_1-**DNE** *and* Π^0_1-**LEM** *are mutually independent.*

The positive parts are done by straightforward logical calculations. The cases of Σ^0_k-**DNE** $\vdash_{\mathsf{HA}}$ Δ^0_k-**LEM**, $(\Pi^0_k \vee \Pi^0_k)$-**DNE** $\vdash_{\mathsf{HA}}$ Δ^0_k-**LEM**, and the equivalence of Σ^0_k-**LLPO**, $B\Sigma^0_{k+1}$-**DNE** and $(\Pi^0_k \vee \Pi^0_k)$-**DNE** are non-trivial.

We sketch a proof of $(\Pi^0_k \vee \Pi^0_k)$-**DNE** $\vdash_{\mathsf{HA}}$ Δ^0_k-**LEM** for $k = 1, 2$. For $k = 1$, assume the equivalence $\forall x.A \Leftrightarrow \exists y.B$, where A and B are atomic formulas. Then, we can easily deduce a contradiction from $\neg(\forall x.A \vee \forall y.\neg B)$. Thus, $\neg\neg(\forall x.A \vee \forall y.\neg B)$ holds. By $(\Pi^0_1 \vee \Pi^0_1)$-**DNE**, we see $\forall x.A \vee \forall y.\neg B$. By the assumption, we see $\exists y.B \vee \neg\exists y.B$.

Note that we used the fact $\neg B$ is of degree 0 to apply $(\Pi^0_1 \vee \Pi^0_1)$-**DNE**. For the case $k = 2$, B is of the form $\forall z.C$. It is easy to see $\neg\forall z.C$ is equivalent to $\exists z.\neg C$ under $(\Pi^0_2 \vee \Pi^0_2)$-**DNE**, since Σ^0_1-**LEM** is derived from $(\Pi^0_2 \vee \Pi^0_2)$-**DNE**. Thus, we can apply the same argument to the case of $k = 2$. The cases of higher degrees are similar.

Σ^0_k-**DNE** $\vdash_{\mathsf{HA}}$ Δ^0_k-**LEM** are proved similarly. The three principles related to LLPO are proved equivalent by the help of the distributivity rule $(\Pi^0_k \vee \Pi^0_k)$-**DNE** $\vdash_{\mathsf{HA}}$ $\forall x_0.\forall x_1.(O_0 \vee O_1) \Leftrightarrow \forall x_0.O_0 \vee \forall x_1.O_1$ for $O_1, O_2 \in \Sigma^0_{k-1}$ proved in Case 2 of the proof of Theorem 2.7. The distributivity rule is a candidate of **LLPO** for the higher degrees.

The difficult parts are underivability results. However, some of the underivability results have straightforward proofs. For example, unprovability of Σ^0_k-**LLPO** from Σ^0_k-**DNE** is done by Kleene realizability with Δ^0_k-functions. For $k = 1$, Σ^0_1-**DNE** is Markov's principle and is realized by the standard Kleene realizability with Δ^0_1-functions, i.e., recursive functions. Thus, if Σ^0_1-**LLPO** is provable from Σ^0_1-**DNE** in HA, then it is also realizable. Take primitive recursive predicates Q_0 and Q_1 so that $P \equiv \exists y.Q_0(x,y)$ and $R \equiv \exists y.Q_1(x,y)$ represent recursively non-separable disjoint enumerably recursive predicates (Theorem II.2.5 [14]). Then, we can easily built a recursive set separating these two disjoint sets from any realizer of the instance of Σ^0_1-**LLPO**. This is a contradiction. For $k > 0$, we can use the same argument by replacing recursive functions by Δ^0_k-functions.

Note that from the underivability proved right above, we can deduce the underivability of Π^0_k-**LEM** from Σ^0_k-**DNE**. Otherwise, Σ^0_k-**LLPO** is derivable from Σ^0_k-**DNE**, since Σ^0_k-**LLPO** is derivable from Π^0_k-**LEM**. Many underivability are derived from in this way from other underivability results.

In the next three subsections, we will sketch three non-trivial unprovability results: Σ^0_n-**LEM** $\nvdash_{\mathsf{HA}}$ Δ^0_{n+1}-**LEM**, Π^0_{n+1}-**LEM** $\nvdash_{\mathsf{HA}}$ Σ^0_{n+1}-**DNE**, and Σ^0_{n+1}-**LLPO** $\nvdash_{\mathsf{HA}}$ Π^0_{n+1}-**LEM**. The three underivability results are proved by various techniques of mathematical logic, such as a cut-free theorem and various realizability interpretations. All of the underivability results of the Main theorem are derived from them and some derivability result. For instance, from Π^0_{n+1}-**LEM** $\nvdash_{\mathsf{HA}}$ Σ^0_{n+1}-**DNE** and Π^0_{n+1}-**LEM** $\vdash_{\mathsf{HA}}$ Δ^0_{n+1}-**LEM** we get Δ^0_{n+1}-**LEM** $\nvdash_{\mathsf{HA}}$ Σ^0_{n+1}-**DNE**.

3.1 Σ^0_n-LEM $\nvdash_{\mathsf{HA}}$ Δ^0_{n+1}-LEM

The underivability of Δ^0_{n+1}-**LEM** from Σ^0_n-**LEM** is proved by a proof theoretic analysis together with a recursion theoretic argument. To prove the underivability, we introduce an infinitary extension $\mathsf{infHA}^{(n)}$ of HA.

In this subsection, we assume that HA is formulated so that it has only successor symbol as its unique function symbol, but has predicate symbols for all primitive recursive relations. See 1.3.6 of [17] for such a formulation of HA. This is for technical simplicity to formulate $\mathsf{infHA}^{(n)}$.

Definition 3.2 ($\mathsf{infHA}^{(n)}$) *The formulas of* $\mathsf{infHA}^{(n)}$ *are the closed formulas of* HA*. The system of* $\mathsf{infHA}^{(n)}$ *is formulated as Gentzen's* LJ*. The initial sequents are the sequents only with atomic formulas that are valid in the standard model* $\mathbb{N}$*. The logical inference rules of* $\mathsf{infHA}^{(n)}$ *are the ones of* LJ *except* $\forall$*-right rule and* $\exists$*-left rule, which are replaced with their infinitary versions (ω-rules). A proof tree of* $\mathsf{infHA}^{(n)}$ *is an infinitary trees labelled with sequents with the following conditions: (i) each node represents a correct inference rule or an initial sequent, (ii) the characteristic function* ϕ *of the proof tree is a* $\emptyset^{(n)}$*-computable function, (iii) the labelled sequent for a node* s *is* $\psi(s)$*, given by a* $\emptyset^{(n)}$*-computable psi.*

Note that $\emptyset^{(n)}$ is the n-th jump [14]. A $\emptyset^{(n)}$-computable function is a function recursive in $\emptyset^{(n)}$. Thus, $\mathsf{infHA}^{(0)}$ has recursively represented infinitary proofs trees. A proof of $\mathsf{infHA}^{(k)}$ is represented by the pair (ϕ, ψ) of $\emptyset^{(k)}$-computable functions ϕ and ψ. By the standard cut-elimination procedure for infinitary proofs, any proof (ϕ, ψ) of $\mathsf{infHA}^{(n)}$ can be transformed into a cut-free form (ϕ', ψ') again of $\mathsf{infHA}^{(n)}$. The translation is $\emptyset^{(n)}$-computable, i.e. identifying $\emptyset^{(n)}$-functions with their indexes, there is a partial recursive function F such that $(\phi', \psi') = F(\phi, \psi)$. These facts are well-known for the case that the proof trees are recursively presented, that is, in the case of $\mathsf{infHA}^{(0)}$. It is very easy to check the same argument remains valid for the general case, if one notices $\emptyset^{(n)}$-computable functions, i.e. Δ^0_{n+1}-functions, satisfy axioms of abstract recursion theories.

Although $\mathsf{infHA}^{(n)}$ has only intuitionistic inference rules, it proves classical logical theorems.

Lemma 3.3 $\mathsf{infHA}^{(n)}$ *proves* Σ^0_n-**LEM**.

We sketch a proof for $n = 1$. Note that Σ^0_n-**LEM** must be a closed formula in this lemma, since $\mathsf{infHA}^{(n)}$ does not have open formulas. Thus, the correct formulation of Σ^0_1-**LEM** is $\forall x.(\exists y.P(x,y) \vee \neg\exists y.P(x,y))$ instead of $\exists y.P(x,y) \vee \neg\exists y.P(x,y)$. By means of the first jump $\emptyset^{(1)}$, we can compute if $\exists y.P(\underline{x},y)$ is correct or not for each x. $\underline{x}$ is the numeral for the integer x. If $\exists y.P(\underline{x},y)$ is correct, we can $\emptyset^{(0)}$-compute y so that $P(\underline{x},\underline{y})$ is provable in $\mathsf{infHA}^{(0)}$. Thus, we can $\emptyset^{(0)}$-compute an $\mathsf{infHA}^{(0)}$-proof of $\exists y.P(\underline{x},y)$. If it is incorrect, we can $\mathsf{infHA}^{(0)}$-compute an $\mathsf{infHA}^{(0)}$-proof of the sequent $P(\underline{x},\underline{y}) \Rightarrow$ for any y. Thus, we can $\mathsf{infHA}^{(0)}$-compute an $\mathsf{infHA}^{(0)}$-proof of the sequent $\Rightarrow \forall y.P(\underline{x},y)$. By these two facts, $\mathsf{infHA}^{(0)}$-proof of $\exists y.P(\underline{x},y) \vee \neg\exists y.P(\underline{x},y)$ is $\emptyset^{(1)}$-computable from x. Thus, $\forall x.(\exists y.P(x,y) \vee \neg\exists y.P(x,y))$ is provable in $\mathsf{infHA}^{(1)}$. The higher cases are proved similarly.

Assume that Σ^0_n-**LEM** $\vdash_{\mathsf{HA}}$ Δ^0_{n+1}-**LEM**. Since all provable closed formulas of HA are again proved in $\mathsf{infHA}^{(n)}$, $\mathsf{infHA}^{(n)}$ proves Δ^0_{n+1}-**LEM** by Lemma 3.3. Note that we can $\emptyset^{(n)}$-compute an $\mathsf{infHA}^{(n)}$-proof of Δ^0_{n+1}-**LEM** from the formula of Δ^0_{n+1}-**LEM**. We will deduce a contradiction from this fact.

Let $X \subseteq \mathbb{N}$ be any $\emptyset^{(n)}$-computable set. By means of some quantifier-free formulas P, Q in the language of $\mathsf{infHA}^{(n)}$, it is written as

$$X = \{i \mid \mathbb{N} \models \exists x.\, P(x,\underline{i})\} = \{i \mid \mathbb{N} \models \forall x.\, Q(x,\underline{i})\}. \quad (1)$$

X is said to have $c \in \mathbb{N}$ as a $\emptyset^{(n)}$-*characteristic index*, if the characteristic function of X is a $\emptyset^{(n)}$-computable function of the index c.

When we encode $\emptyset^{(n)}$-computable sets by $\emptyset^{(n)}$-characteristic indexes, then the standard diagonalization argument shows that the class $\mathcal{X}$ of all $\emptyset^{(n)}$-computable sets is *not* $\emptyset^{(n)}$-effectively enumerable.

However, by assuming $\mathsf{infHA}^{(n)}$ proves Δ^0_{n+1}-**LEM**, we can construct a $\emptyset^{(n)}$-effective enumeration of the class $\mathcal{X}$. The proof is by the following lemma, which is proved by an analysis of cut-free forms of proofs of $(A_1 \Leftrightarrow A_2) \supset C_1 \vee C_2$ in $\mathsf{infHA}^{(n)}$.

Lemma 3.4 *Suppose a formula* $(A_1 \Leftrightarrow A_2) \supset C_1 \vee C_2$ *is* $\mathsf{infHA}^{(n)}$*-provable. Then an* $\mathsf{infHA}^{(n)}$*-proof of one of the following formulas is* $\emptyset^{(n)}$*-computable: (a)* $(A_1 \Leftrightarrow A_2) \supset C_1$*, (b)* $(A_1 \Leftrightarrow A_2) \supset C_2$*, (c)* $(A_1 \Leftrightarrow A_2) \supset A_1$*, (d)* $(A_1 \Leftrightarrow A_2) \supset A_2$*.*

Assume $\mathsf{infHA}^{(n)}$ proves the following Δ^0_{n+1}-**LEM**:

$$\forall z.((\exists x.\, P(x,z) \Leftrightarrow \forall x.\, Q(x,z)) \supset \exists x.\, P(x,z) \vee \neg\exists x.\, P(x,z)).$$

By the lemma introduced above, we can $\emptyset^{(n)}$-compute from i one of the following:

$$(\exists x.\, P(x,\underline{i}) \Leftrightarrow \forall x.\, Q(x,\underline{i})) \supset \exists x.\, P(x,\underline{i}),$$
$$(\exists x.\, P(x,\underline{i}) \Leftrightarrow \forall x.\, Q(x,\underline{i})) \supset \neg\exists x.\, P(x,\underline{i}).$$

Since $\mathsf{infHA}^{(n)}$ is sound to the standard model of PA and by the condition (1) on P and Q stated above, one of $\exists x.\, P(x,\underline{i})$ and $\neg\exists x.\, P(x,\underline{i})$ is true. In the former case, $i \in X$ holds, and in the latter case, $i \notin X$ holds. Thus, we can $\emptyset^{(n)}$-decide whether $i \in X$ or not. By this fact, we can easily construct a total $\emptyset^{(n)}$-function $\Phi(e,i)$ enumerating $\mathcal{X}$ such that (i) $\{i \mid \Phi(e,i) = 0\}$ is a $\emptyset^{(n)}$-computable set, (ii) for any $\emptyset^{(n)}$-computable set X, there is e such that $X = \{i \mid \Phi(e,i) = 0\}$.

The construction of Φ is as follows: Since all proofs of Δ^0_{n+1}-**LEM** from Σ^0_n-**LEM** in HA are $\emptyset^{(0)}$-enumerable, we can $\emptyset^{(0)}$-enumerate $\mathsf{infHA}^{(n)}$-proofs of all (universally closed) instances of Δ^0_{n+1}-**LEM** from Σ^0_n-**LEM**. Since the proofs of Σ^0_n-**LEM** of $\mathsf{infHA}^{(n)}$ are $\emptyset^{(n)}$-enumerable, we can $\emptyset^{(n)}$-enumerate proofs of all Δ^0_{n+1}-**LEM**. Thus, from a coding e of predicates P and Q, we can $\emptyset^{(n)}$-compute a $\mathsf{infHA}^{(n)}$-proof of the corresponding Δ^0_{n+1}-**LEM**. By the decision method given above, we can $\emptyset^{(n)}$-decide if $i \in X$. This $\emptyset^{(n)}$-algorithm gives some Φ enumerating $\emptyset^{(n)}$-computable sets. Contradiction.

3.2 Π^0_{n+1}-LEM $\nvdash_{\mathsf{HA}}$ Σ^0_{n+1}-DNE

The proof of the unprovability Π^0_n-**LEM** $\nvdash_{\mathsf{HA}}$ Σ^0_n-**DNE** is based on the monotone modified realizability in [11]. For technical reasons, we give the interpretation not for HA but for the extensional finite-type arithmetic $\mathsf{E\text{-}HA}^{\omega,(n)}$ with $\emptyset^{(n)}$-oracles, which is an extensional variant $\mathsf{E\text{-}HA}^{\omega}$ of intuitionistic arithmetic in all finite types [17, Sect. 1.6.12]. We will use Greek letters for variables, and letters $x_1, \ldots, x_k, y, z, \ldots$ for variables of type 0.

Definition 3.5 ($\mathsf{E\text{-}HA}^{\omega,(n)}$) *An* extensional finite-type arithmetic $\mathsf{E\text{-}HA}^{\omega,(n)}$ with $\emptyset^{(n)}$-oracles *is defined by induction on n.* $\mathsf{E\text{-}HA}^{\omega,(0)}$ *is* $\mathsf{E\text{-}HA}^{\omega}$*.* $\mathsf{E\text{-}HA}^{\omega,(n+1)}$ *is the extension of* $\mathsf{E\text{-}HA}^{\omega,(n)}$ *with new function constants* $f_{\exists y P(x_1,\ldots,x_k,y)}$ *for all quantifier-free formulas* $P(x_1,\ldots,x_k,y)$ *of* $\mathsf{E\text{-}HA}^{\omega,(n)}$*, where* $x_1,\ldots,x_k,y$ *are of type 0. The axiom*

$$P(x_1,\ldots,x_k,y) \supset P(x_1,\ldots,x_k, f_{\exists y P(x_1,\ldots,x_k,y)} x_1 \ldots x_k)$$

is added for the new constant.

In the following we implicitly refer to the obvious embedding of HA into $\mathsf{E\text{-}HA}^{\omega,(0)}$ (see e.g. [12], 1.6.9). We consider the following principles:

$$\begin{array}{ll} E^2 & \exists \Phi^{(N\to N)\to N} \forall \xi^{N\to N}. \\ & \quad \left[\Phi^{(N\to N)\to N}\xi^{N\to N} = 0 \Leftrightarrow \forall x.\, \xi x = 0\right] \\ AC & \forall \xi^\sigma \exists \eta^\tau.\, A(\xi,\eta) \supset \exists \varphi^{\sigma\to\tau} \forall \xi^\sigma.\, A(\xi, \varphi\xi) \end{array}$$

Since $\mathsf{E\text{-}HA}^{\omega,(n)}$ has constants and axioms for n-th jump, we easily see the following lemma.

Lemma 3.6 *Let $O(x_1,\dots,x_k)$ be any Σ^0_n-formula in the language of* HA. *Then there is a quantifier-free formula $R(x_1,\dots,x_k)$ in the language of $\mathsf{E\text{-}HA}^{\omega,(n)}$ such that $\mathsf{E\text{-}HA}^{\omega,(n)}$ proves $O(x_1,\dots,x_k) \Leftrightarrow R(x_1,\dots,x_k)$. The same result holds for Π^0_n-formula $C(x_1,\dots,x_k)$ in the language of* HA .

In $\mathsf{E\text{-}HA}^{\omega,(n)}$, one easily defines the characteristic function of Σ^0_n-sentences in the language of HA. Together with the use of E^2 this gives:

Theorem 3.7 $\mathsf{E\text{-}HA}^{\omega,(n)} + E^2$ *proves every instance of Π^0_{n+1}-**LEM** in the language of* HA.

Note that $\mathsf{E\text{-}HA}^{\omega,(n)}$ is sound in the full type structure $\mathcal{S}$, which is the standard set theoretical model of higher order functions. On the other hand, $\mathsf{E\text{-}HA}^{\omega}$ is sound in the substructure given by the Kleene's S1-S9 computable functionals in [9], [17, pp.162–163]. Furthermore, the new constants $f_{\exists y P(x_1,\dots,x_k,y)}$ can be interpreted by functions recursive in the jump $\emptyset^{(n)}$. Thus, the following lemma holds.

Lemma 3.8 (Calculability) *For any closed term t of $\mathsf{E\text{-}HA}^{\omega,(n)}$ of the type $\overbrace{0 \to (\cdots \to (0}^{k} \to 0))$, t represents a $\emptyset^{(n)}$-computable function from $\mathbb{N}^k$ to $\mathbb{N}$ in the full structure $\mathcal{S}$.*

3.2.1 Monotone modified realizability interpretation.

$\mathsf{E\text{-}HA}^{\omega,(n)} + AC + E^2$ is sound w.r.t. the monotone modified realizability interpretation of [11]. The interpretation is a combination of Howard's notion of majorizability and Kreisel's modified realizability interpretation.

Definition 3.9 (Majorizability) *(W.A. Howard, cf. [17]) The majorizability relation "ξ maj_ρ ζ" is defined by induction on ρ as follows:*

$$\begin{array}{lll} x\ maj_0\ x' & \Leftrightarrow & x \geq x', \\ \xi\ maj_{\rho\to\tau}\ \xi' & \Leftrightarrow & \forall \eta,\eta'(\eta\ maj_\rho\ \eta' \supset \xi\eta\ maj_\tau\ \xi'\eta'). \end{array}$$

If ξ_i maj_{τ_i} ζ_i for each $i \in \{1,\dots,n\}$, then we write $(\xi_1,\dots,\xi_n)$ maj $(\zeta_1,\dots,\zeta_n)$.

Let $\vec{\zeta}$ **mr** A be Kreisel's modified realizability interpretation of A by the sequence of variables $\vec{\zeta}$ (see [17] for the definition). Then the monotone modified realizability interpretation of a $\mathsf{E\text{-}HA}^{\omega,(n)}$-formula A by the sequence of terms $\vec{t}$ is

$$\exists \vec{\zeta}\Big[\vec{t}\ maj\ \vec{\zeta}\ \wedge\ \forall \xi\big(\vec{\zeta}\vec{\xi}\ \mathbf{mr}\ A(\vec{\xi})\big)\Big].$$

Theorem 3.10 $\mathsf{E\text{-}HA}^{\omega,(n)} + AC + E^2$ *is sound with the monotone modified realizability interpretation in* $\mathsf{E\text{-}HA}^{\omega,(n)} + E^2$.

Proof. For $n = 0$ the theorem is proved in [11]. For $n > 0$ one only has to observe that the new function constants can easily be majorized with the help of the functional $\Phi_{\max}(f,x) = \max(f(0),\dots,f(x))$ in $\mathsf{E\text{-}HA}^{\omega,(n)}$. □

This theorem means that if $\mathsf{E\text{-}HA}^{\omega,(n)} + AC + E^2$ proves a formula $A(\vec{\xi})$, then there exists a sequence $\vec{t}$ of closed $\mathsf{E\text{-}HA}^{\omega,(n)}$-terms such that the monotone modified realizability interpretation of A by $\vec{t}$ is provable in $\mathsf{E\text{-}HA}^{\omega,(n)} + E^2$.

Theorem 3.11 *There is a quantifier-free* HA*-formula $P(z,x_1,x_2,\dots,x_{n+1})$ for which Σ^0_{n+1}-**DNE***

$$\begin{array}{l} \neg\neg\, \exists x_1 \forall x_2 \dots Q x_{n+1}.\, P(z,x_1,x_2,\dots,x_{n+1}) \\ \quad \supset \exists x_1 \forall x_2 \dots Q x_{n+1}.\, P(z,x_1,x_2,\dots,x_{n+1}). \end{array}$$

is not provable in $\mathsf{E\text{-}HA}^{\omega,(n)} + AC + E^2$.

Proof. Let us prove for the case $n = 1$. The general case is similarly proved. Take any HA-formula P so that the predicate $\exists x_1 \forall x_2 P(\underline{i}, x_1, x_2)$ is not $\emptyset^{(1)}$-recursive.

Assume that the Σ^0_{n+1}-**DNE** of the theorem is provable for this P. We will deduce a contradiction. By Lemma 3.6, $\mathsf{E\text{-}HA}^{\omega,(1)} + AC + E^2$ proves

$$\forall x_2.P(z,x_1,x_2) \Leftrightarrow U(z,x_1)$$

for some quantifier-free $U(z,x) \in \mathsf{E\text{-}HA}^{\omega,(1)}$. Thus, by the assumption, $\mathsf{E\text{-}HA}^{\omega,(1)} + AC + E^2$ proves

$$\neg\neg\, \exists x.\, U(z,x) \supset \exists x.\, U(z,x).$$

By Theorem 3.10, the monotone modified realizability interpretation of this formula is provable in $\mathsf{E\text{-}HA}^{\omega,(1)} + E^2$ for a closed $\mathsf{E\text{-}HA}^{\omega,(1)}$-term t. Thus, the following is also provable in the system:

$$\exists \zeta^1\Big(t\ maj_1\ \zeta^1\ \wedge\ \forall z.\big(\neg\neg\, \exists x.\, U(z,x)\ \supset\ U(z,\zeta^1 z)\big)\Big).$$

Thus, so does

$$\forall z.\exists x \le tz.\left(\neg\neg\,\exists x.\,U(z,x)\ \supset\ U(z,x)\right).$$

Thus, $\mathsf{E\text{-}HA}^{\omega,(1)} + E^2$ proves

$$\forall z\left(\exists x.\,U(z,x) \Leftrightarrow (\exists x \le tz)\,U(z,x)\right).$$

In the full type structure $\mathcal{S}$, the left-hand side $\exists x.\,U(z,x)$ is not $\emptyset^{(1)}$-recursive. However, the right-hand side $(\exists x \le tz)U(z,x)$ is $\emptyset^{(1)}$-recursive by Lemma 3.8. Contradiction! □

Corollary 3.12 *(i)* Π^0_n-**LEM** $\nvdash_{\mathsf{HA}}$ Σ^0_n-**DNE** *and (ii)* Π^0_n-**LEM** $\nvdash_{\mathsf{HA}}$ Σ^0_n-**LEM** *for non-negative integer n.*

Remark 3.13 As the proof of the previous corollary shows, the underivability of Σ^0_n-**DNE** from Π^0_n-**LEM** over HA even holds over $\mathsf{E\text{-}HA}^{\omega,(n)}$ plus any further principles A which have a computable monotone mr-interpretation, as e.g. AC, (E^2) or even full comprehension for exist-free formulas in all types (see [11]).

3.3 Σ^0_{n+1}-LLPO $\nvdash_{\mathsf{HA}}$ Π^0_{n+1}-LEM

Theorem 3.14 Σ^0_{n+1}-**LLPO** $\nvdash_{\mathsf{HA}}$ Π^0_{n+1}-**LEM**.

Proof. We verify only the case of $n = 0$. The proof remains valid for the general case by replacing the recursive functions by the Δ^0_n-functions. We show that the Lifschitz' realizability interpretation [13] satisfies Σ^0_1-**LLPO** but not Π^0_1-**LEM**. Here, "a number x Lifschitz-realizes a formula A" (x **lr** A in symbol) is defined in the same way as the recursive realizability interpretation except that A is an existential quantification or a disjunction. For example, x **lr** $\exists y.\,A(y)$ is defined by

$$V_x \ne \emptyset\ \wedge\ \forall g \in V_x.\left[j_2(g)\ \mathbf{lr}\ A\big(j_1(g)\big)\right].$$

where V_x is the set $\{n \le j_2(x) \mid j_1(x) \bullet n \uparrow\}$. See [18] for details.

In order to show that the Lifschitz realizability interpretation satisfies Σ^0_1-**LLPO**, it is sufficient to show that the HA-provably equivalent $B\Sigma^0_2$-**DNE** is realizable. In [19, pp.810-811], van Oosten defined $B\Sigma^0_2$-*negative formulas* and proved that they are the "self-realizing" formulas for the Lifschitz' realizability interpretation. Because $B\Sigma^0_2$-**DNE** is a $B\Sigma^0_2$-negative formula, the Lifschitz realizability interpretation satisfies $B\Sigma^0_2$-**DNE**.

On the other hand, the Lifschitz' realizability interpretation [13] does not satisfy Π^0_1-**LEM**. As pointed out in [13], a Lifschitz-realization of the unique existence $\exists !x.A$ implies the existence of a recursive function computing x. Thus, Lifschitz-realization of $\forall y.\,\neg T(x,x,y)\ \vee \neg\forall y.\,\neg T(x,x,y)$ implies a solution of the halting problem. Contradiction. □

There are two other proofs for the same unprovability result. One is by the standard Kleene realizability but with realizers recursive in the model W of the formal system WKL_0 for the weak König lemma constructed in [15]. This one and the proof presented above are the essentially same and their learning theoretic meaning are characterized by "Popperian game" introduced in [7]. A Popperian game is a competition of finite "refutable" theories, i.e., Π^0_1-propositions, and embodies the computational aspects of the low basis theorem in recursion theory. Many theorems in the formal theory WKL_0 of [15], e.g., the completeness theorem of the classical predicate logic, can be proved in HA^ω + **WKL**, and their computational contents can be extracted as Popperian games. These facts strongly suggest a strict relationship between mathematics using only semi-classical principles and Reverse Mathematics in [15] (cf. [16]).

Another sharply different and proof theoretically more sensible proof is by the monotone functional interpretation introduced in [10]. A detailed proof for $n = 1$ of actually much stronger results will be found in Corollary 8.11 and the subsequent discussion of [12].

These things will be discussed with detailed proofs in our forthcoming paper(s) on calibrations of mathematical theorems by means of semi-classical principles.

4 Conclusion

We proved the existence of a hierarchy, from the intuitionistic viewpoint, between relevant semi-classical principles. In particular, we proved that Limited Principle of Omniscience, and Markov's principle (even taken together), are but a proper part of Excluded Middle for degree 1 formulas, or for simply universal formulas. This means that a proof using only degree 1 Limited Principle Omniscience, or Markov's Principle, or both, works on strictly weaker assumptions than a proof using Excluded Middle, say, for degree 1 formulas. This provides a theoretical background for a difference we did already known. In fact, if we use the two former principles, we are able to gather concrete information from a proof, like extraction of effective bounds, or a simpler interpretation in term of learning. This is something which still remains true for Π^0_1-**LEM** alone, provided we drop Markov's principle ([11], [12]), but which definitively fails for Σ^0_1-**LEM**.

A similar remark holds for our results on "constructive" principles (principles which allow to extract skolem maps out of proofs of $\forall\exists$-statements). We proved that Post's Theorem is not intuitionistically provable. Yet, Post's Theorem is provable from Markov's principle. This means that there are results of constructive mathematics which are not intuitionistically provable, like Markov's principle, yet which are intuitionistically weaker than this latter. This fact is somehow puzzling, and we still miss a convincing interpretation for it. Remark that the nature of the difference between Markov's principle and intuitionism is, instead, something well-known. We have an intuitive interpretation of Markov Principle in term of blind search algorithm. Then any proof of a statement $A = \forall x.\exists y.P(x, y)$ using Markov principle provides a recursive f such that $P(x, f(x))$ for all x, but it does not describe *explicitly* a bound for f. (Of course, some information about bounds may still be extracted from the proof of A using Proof Theory). Does some difference of this kind exist if we derive A from Post's Theorem?

We end with a conjecture, due to one author: there are results in Constructive Analysis which are, from an intuitionistic viewpoint, strictly between Post's Theorem and Markov's principle. Maybe, there is an entire hierarchy inside constructivism waiting to be discovered, and, above all, understood.

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