Effective Borel Measurability and Reducibility of Functions

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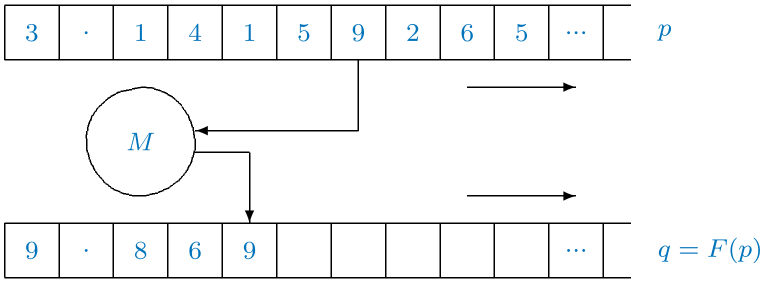
Contents

Turing machines

Definition 1 A function $F :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ is called *computable*, if there exists a Turing machine with one-way output tape which transfers each input $p \in \mathrm{dom}(F)$ into the corresponding output $F(p)$.

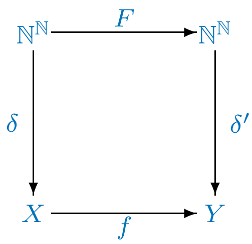


Proposition 2 *Any computable function $F :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ is continuous with respect to the Baire topology on $\mathbb{N}^\mathbb{N}$.*

Computable functions

Definition 3 A *representation* of a set X is a surjective function $\delta :\subseteq \mathbb{N}^\mathbb{N} \to X$.

Definition 4 A function $f :\subseteq X \rightrightarrows Y$ is called *(δ, δ')–computable*, if there exists a computable function $F :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ such that $\delta' F(p) \in f\delta(p)$ for all $p \in \mathrm{dom}(f\delta)$.



Definition 5 If δ, δ' are admissible representations of topological spaces X, Y, respectively, then there is a canonical representation $[\delta \to \delta']$ of $\mathcal{C}(X, Y) := \{f : X \to Y : f \text{ continuous}\}$.

Computable metric spaces

Definition 6 A tuple (X, d, α) is called a *computable metric space*, if

1. $d : X \times X \to \mathbb{R}$ is a metric on X,
2. $\alpha : \mathbb{N} \to X$ is a sequence which is dense in X,
3. $d \circ (\alpha \times \alpha) : \mathbb{N}^2 \to \mathbb{R}$ is a computable (double) sequence in $\mathbb{R}$.

Definition 7 Let (X, d, α) be a computable metric space. The *Cauchy representation* $\delta_X :\subseteq \mathbb{N}^{\mathbb{N}} \to X$ of X is defined by

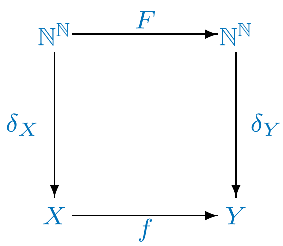
$$\delta_X(p) := \lim_{i \to \infty} \alpha p(i)$$

for all p such that $(\alpha p(i))_{i \in \mathbb{N}}$ converges and $d(\alpha p(i), \alpha p(j)) < 2^{-i}$ for all $j > i$ (and undefined for all other input sequences).

Kreitz-Weihrauch Representation Theorem

Theorem 8 *Let X, Y be computable metric spaces and let $f :\subseteq X \to Y$ be a function. Then the following are equivalent:*

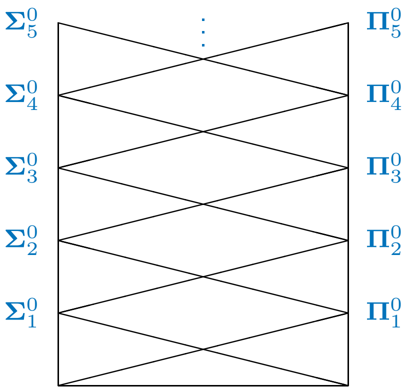
1. *f is continuous,*
2. *f admits a continuous realizer $F :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$.*



Question: Can this theorem be generalized to Borel measurable functions?

Borel hierarchy

- $\mathbf{\Sigma}^0_1(X)$ is the set of open subsets of X,
- $\mathbf{\Pi}^0_1(X)$ is the set of closed subsets of X,
- $\mathbf{\Sigma}^0_2(X)$ is the set of F_σ subsets of X,
- $\mathbf{\Pi}^0_2(X)$ is the set of G_δ subsets of X, etc.
- $\mathbf{\Delta}^0_k(X) := \mathbf{\Sigma}^0_k(X) \cap \mathbf{\Pi}^0_k(X)$.



Representations of Borel classes

Definition 9 Let (X, d, α) be a separable metric space. We define representations $\delta_{\mathbf{\Sigma}^0_k(X)}$ of $\mathbf{\Sigma}^0_k(X)$, $\delta_{\mathbf{\Pi}^0_k(X)}$ of $\mathbf{\Pi}^0_k(X)$ and $\delta_{\mathbf{\Delta}^0_k(X)}$ of $\mathbf{\Delta}^0_k(X)$ for $k \geq 1$ as follows:

- $\delta_{\mathbf{\Sigma}^0_1(X)}(p) := \bigcup_{\langle i,j\rangle \in \mathrm{range}(p)} B(\alpha(i), \overline{j})$,
- $\delta_{\mathbf{\Pi}^0_k(X)}(p) := X \setminus \delta_{\mathbf{\Sigma}^0_k(X)}(p)$,
- $\delta_{\mathbf{\Sigma}^0_{k+1}(X)}\langle p_0, p_1, \ldots\rangle := \bigcup_{i=0}^{\infty} \delta_{\mathbf{\Pi}^0_k(X)}(p_i)$,
- $\delta_{\mathbf{\Delta}^0_k(X)}\langle p, q\rangle = \delta_{\mathbf{\Sigma}^0_k(X)}(p) :\iff \delta_{\mathbf{\Sigma}^0_k(X)}(p) = \delta_{\mathbf{\Pi}^0_k(X)}(q)$,

for all $p, p_i, q \in \mathbb{N}^{\mathbb{N}}$.

Effective closure properties of Borel classes

Proposition 10 *Let X, Y be computable metric spaces. The following operations are computable for any $k \geq 1$:*

1. $\mathbf{\Sigma}^0_k \hookrightarrow \mathbf{\Sigma}^0_{k+1}$, $\mathbf{\Sigma}^0_k \hookrightarrow \mathbf{\Pi}^0_{k+1}$, $\mathbf{\Pi}^0_k \hookrightarrow \mathbf{\Sigma}^0_{k+1}$, $\mathbf{\Pi}^0_k \hookrightarrow \mathbf{\Pi}^0_{k+1}$, $A \mapsto A$ *(injection)*
2. $\mathbf{\Sigma}^0_k \to \mathbf{\Pi}^0_k$, $\mathbf{\Pi}^0_k \to \mathbf{\Sigma}^0_k$, $A \mapsto A^{\mathrm{c}} := X \setminus A$ *(complement)*
3. $\mathbf{\Sigma}^0_k \times \mathbf{\Sigma}^0_k \to \mathbf{\Sigma}^0_k$, $\mathbf{\Pi}^0_k \times \mathbf{\Pi}^0_k \to \mathbf{\Pi}^0_k$, $(A, B) \mapsto A \cup B$ *(union)*
4. $\mathbf{\Sigma}^0_k \times \mathbf{\Sigma}^0_k \to \mathbf{\Sigma}^0_k$, $\mathbf{\Pi}^0_k \times \mathbf{\Pi}^0_k \to \mathbf{\Pi}^0_k$, $(A, B) \mapsto A \cap B$ *(intersection)*
5. $(\mathbf{\Sigma}^0_k)^{\mathbb{N}} \to \mathbf{\Sigma}^0_k$, $(A_n)_{n\in\mathbb{N}} \mapsto \bigcup_{n=0}^{\infty} A_n$ *(countable union)*
6. $(\mathbf{\Pi}^0_k)^{\mathbb{N}} \to \mathbf{\Pi}^0_k$, $(A_n)_{n\in\mathbb{N}} \mapsto \bigcap_{n=0}^{\infty} A_n$ *(countable intersection)*
7. $\mathbf{\Sigma}^0_k(X) \times \mathbf{\Sigma}^0_k(Y) \to \mathbf{\Sigma}^0_k(X \times Y)$, $(A, B) \mapsto A \times B$ *(product)*
8. $(\mathbf{\Pi}^0_k(X))^{\mathbb{N}} \to \mathbf{\Pi}^0_k(X^{\mathbb{N}})$, $(A_n)_{n\in\mathbb{N}} \mapsto \times_{n=0}^{\infty} A_n$ *(countable product)*
9. $\mathbf{\Sigma}^0_k(X \times \mathbb{N}) \to \mathbf{\Sigma}^0_k(X)$, $A \mapsto \{x \in X : (\exists n)(x, n) \in A\}$ *(countable projection)*
10. $\mathbf{\Sigma}^0_k(X \times Y) \times Y \to \mathbf{\Sigma}^0_k(X)$, $(A, y) \mapsto A_y := \{x \in X : (x, y) \in A\}$ *(section)*

Borel measurable operations

Definition 11 Let X, Y be separable metric spaces. A multi-valued operation $f : X \rightrightarrows Y$ is called

- $\mathbf{\Sigma}^0_k$*–measurable*, if $f^{-1}(U) \in \mathbf{\Sigma}^0_k(X)$ for any $U \in \mathbf{\Sigma}^0_1(Y)$,
- *effectively* $\mathbf{\Sigma}^0_k$*–measurable* or $\mathbf{\Sigma}^0_k$*–computable*, if the map

  $$\mathbf{\Sigma}^0_k(f^{-1}) : \mathbf{\Sigma}^0_1(Y) \to \mathbf{\Sigma}^0_k(X), U \mapsto f^{-1}(U)$$

  is computable.

Definition 12 Let X, Y be separable metric spaces. We define representations $\delta_{\mathbf{\Sigma}^0_k(X \rightrightarrows Y)}$ of $\mathbf{\Sigma}^0_k(X \rightrightarrows Y)$ by

$$\delta_{\mathbf{\Sigma}^0_k(X \rightrightarrows Y)}(p) = f :\iff [\delta_{\mathbf{\Sigma}^0_1(Y)} \to \delta_{\mathbf{\Sigma}^0_k(X)}](p) = \mathbf{\Sigma}^0_k(f^{-1})$$

for all $p \in \mathbb{N}^{\mathbb{N}}$, $f : X \rightrightarrows Y$ and $k \geq 1$. Let $\delta_{\mathbf{\Sigma}^0_k(X \to Y)}$ denote the restriction to $\mathbf{\Sigma}^0_k(X \to Y)$.

Effective closure properties of Borel measurable operations

Proposition 13 *Let W, X, Y and Z be computable metric spaces. The following operations are computable for all $n, k \geq 1$:*

1. $\mathbf{\Sigma}^0_n(Y \rightrightarrows Z) \times \mathbf{\Sigma}^0_k(X \to Y) \to \mathbf{\Sigma}^0_{n+k-1}(X \rightrightarrows Z), (g, f) \mapsto g \circ f$ *(composition)*
2. $\mathbf{\Sigma}^0_k(X \rightrightarrows Y) \times \mathbf{\Sigma}^0_k(X \rightrightarrows Z) \to \mathbf{\Sigma}^0_k(X \rightrightarrows Y \times Z), (f, g) \mapsto (x \mapsto f(x) \times g(x))$ *(juxtaposition)*
3. $\mathbf{\Sigma}^0_k(X \rightrightarrows Y) \times \mathbf{\Sigma}^0_k(W \rightrightarrows Z) \to \mathbf{\Sigma}^0_k(X \times W \rightrightarrows Y \times Z), (f, g) \mapsto f \times g$ *(product)*
4. $\mathbf{\Sigma}^0_k(X \rightrightarrows Y^{\mathbb{N}}) \to \mathbf{\Sigma}^0_k(X \times \mathbb{N} \rightrightarrows Y), f \mapsto f_*$ *(evaluation)*
5. $\mathbf{\Sigma}^0_k(X \times \mathbb{N} \rightrightarrows Y) \to \mathbf{\Sigma}^0_k(X \rightrightarrows Y^{\mathbb{N}}), f \mapsto [f]$ *(transposition)*
6. $\mathbf{\Sigma}^0_k(X \rightrightarrows Y) \to \mathbf{\Sigma}^0_k(X^{\mathbb{N}} \rightrightarrows Y^{\mathbb{N}}), f \mapsto f^{\mathbb{N}}$ *(exponentiation)*
7. $\mathbf{\Sigma}^0_k(X \times \mathbb{N} \rightrightarrows Y) \to \mathbf{\Sigma}^0_k(X \rightrightarrows Y)^{\mathbb{N}}, f \mapsto (n \mapsto (x \mapsto f(x, n)))$ *(sequencing)*
8. $\mathbf{\Sigma}^0_k(X \rightrightarrows Y)^{\mathbb{N}} \to \mathbf{\Sigma}^0_k(X \times \mathbb{N} \rightrightarrows Y), (f_n)_{n \in \mathbb{N}} \mapsto ((x, n) \mapsto f_n(x))$ *(de-sequencing)*

Composition

Proof. For all $U \in \mathbf{\Sigma}^0_1(Z)$ and $A_i \in \mathbf{\Pi}^0_{n-1}(Y)$ with $g^{-1}(U) = \bigcup_{i=0}^{\infty} A_i$ we obtain in case $n > 1$

1. $(g \circ f)^{-1}(U) = f^{-1}g^{-1}(U)$,
2. $f^{-1}(\bigcup_{i=0}^{\infty} A_i) = \bigcup_{i=0}^{\infty} f^{-1}(A_i)$,
3. $f^{-1}(A_i) = X \setminus f^{-1}(Y \setminus A_i)$.

Since $f^{-1}(Y \setminus A_i) \in \mathbf{\Sigma}^0_{n+k-2}(X)$ we obtain $(g \circ f)^{-1}(U) = \bigcup_{i=0}^{\infty} f^{-1}(A_i) \in \mathbf{\Sigma}^0_{n+k-1}(X)$. □

Corollary 14 *Let X, Y and Z be computable metric spaces and $n, k \in \mathbb{N}$. If $f : X \to Y$ is $\mathbf{\Sigma}^0_{n+1}$–computable and $g : Y \rightrightarrows Z$ is $\mathbf{\Sigma}^0_{k+1}$–computable, then $g \circ f$ is $\mathbf{\Sigma}^0_{n+k+1}$–computable.*

(In case of $n = 1$ the same holds for multi-valued $f : X \rightrightarrows Y$).

Uniform convergence

Proposition 15 *Let X, Y be computable metric spaces and $k \geq 1$. The following operation is computable:*

$$\mathrm{Lim} :\subseteq \mathbf{\Sigma}^0_k(X \rightrightarrows Y)^{\mathbb{N}} \to \mathbf{\Sigma}^0_k(X \to Y),$$
$$(f_n)_{n\in\mathbb{N}} \mapsto (x \mapsto \{\lim_{n\to\infty} y_n : y_n \in f_n(x)\}),$$

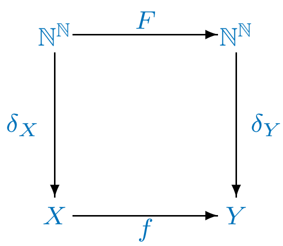
defined for all sequences $(f_n)_{n\in\mathbb{N}}$ of $\mathbf{\Sigma}^0_k$–measurable multi-valued functions $f_n : X \rightrightarrows Y$ which fulfill $d(y_i, y_j) < 2^{-j}$ for all $x \in X$ and $i > j$ where $y_n \in f_n(x)$ and any such sequence $(y_n)_{n\in\mathbb{N}}$ is convergent.

Corollary 16 *Let X, Y be computable metric spaces and $k \geq 1$. If $(f_n)_{n\in\mathbb{N}}$ is a computable and pointwise convergent sequence of $\mathbf{\Sigma}^0_k$–computable functions $f_n : X \to Y$ such that additionally $d(f_i(x), f_j(x)) < 2^{-j}$ for all $x \in X$ and $i > j$, then the limit function $f : X \to Y$ is $\mathbf{\Sigma}^0_k$–computable as well.*

Representation Theorem

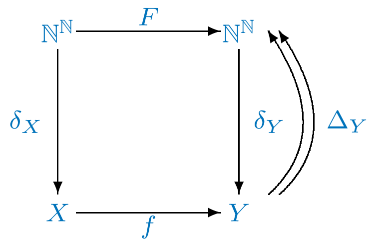
Theorem 17 *Let X, Y be computable metric spaces, $k \geq 1$ and let $f : X \to Y$ be a total function. Then the following are equivalent:*

1. *f is (effectively) $\mathbf{\Sigma}^0_k$–measurable,*
2. *f admits an (effectively) $\mathbf{\Sigma}^0_k$–measurable realizer $F :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$.*



Proof of 1. $\Longrightarrow$ 2.

Proof. Given a $\mathbf{\Sigma}^0_k$–measurable f, we can effectively find a $\mathbf{\Sigma}^0_k$–measurable selector F of the composition $\Delta_Y \circ f \circ \delta_X : \mathrm{dom}(\delta_X) \rightrightarrows Y$ by the effective Kuratowski-Ryll-Nardzewski Selection Theorem.



□

Effective Kuratowski-Ryll-Nardzewski Selection Theorem

Theorem 18 *Let X, Y be computable metric spaces and let Y be complete and $k \geq 2$. There is a computable operation $S : \mathbf{\Sigma}^0_k(X \rightrightarrows Y) \rightrightarrows \mathbf{\Sigma}^0_k(X \to Y)$ such that $f(x) \in \overline{F(x)}$ for any $f \in S(F), F \in \mathbf{\Sigma}^0_k(X \rightrightarrows Y)$ and $x \in X$.*

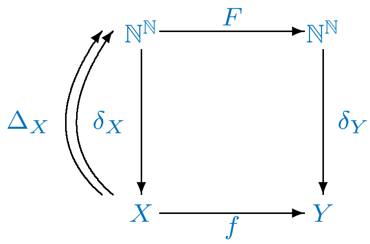
Proof. Given a $\mathbf{\Sigma}^0_k$–measurable operation $F : X \rightrightarrows Y$ we construct a sequence of $\mathbf{\Sigma}^0_k$–measurable mappings $f_n : X \to Y$ which fulfill

$$d_{F(x)}(f_n(x)) < 2^{-n},$$
$$d(f_n(x), f_{n-1}(x)) < 2^{-n}$$

for any $n \in \mathbb{N}$ and we will apply the uniform convergence closure scheme to this sequence. □

Proof idea for 2. $\Longrightarrow$ 1.

Proof. Use the identity $f = \delta_Y \circ F \circ \Delta_X$.



This proof idea only works in case of $k = 1$! The idea can be extended to the case $k = 2$ if δ_X is replaced by some equivalent representation with very well-behaved preimages (Schröder's representation).

□

Schröder's representation

Definition 19 Let (X, d, α) be a separable metric space. Define $\widehat{\sigma}_X :\subseteq \mathbb{N}^{\mathbb{N}} \to X$ by $\widehat{\sigma}_X(p) = x$ if and only if $p \in \{0,1\}^{\mathbb{N}}$ and

$$(\forall i, j \in \mathbb{N}) \begin{cases} p\langle i,j\rangle = 0 \Longrightarrow d(x, \alpha(i)) \leq 2^{-j} \\ p\langle i,j\rangle = 1 \Longrightarrow d(x, \alpha(i)) \geq 2^{-j-1} \end{cases}$$

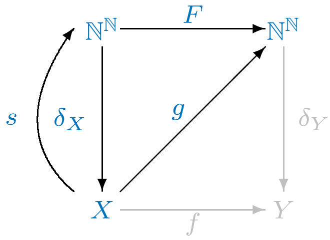
for all $x \in X$ and $p \in \mathbb{N}$.

Lemma 20 *Let X be a computable metric space. Then:*

1. $\widehat{\sigma}_X \equiv_c \delta_X$.
2. $\widehat{\kappa}_X : \mathcal{K}_>(X) \to \mathcal{K}_>(\mathbb{N}^{\mathbb{N}}), K \mapsto \widehat{\sigma}_X^{-1}(K)$ *is computable.*
3. $\Phi_X : \Gamma(\mathbb{N}^{\mathbb{N}}) \to \Gamma(X), A \mapsto \widehat{\sigma}_X(A)$ *is computable for* $\Gamma \in \{\mathbf{\Pi}^0_1, \mathbf{\Sigma}^0_2\}$.
4. $\widehat{\Delta}_X : X \rightrightarrows \mathbb{N}^{\mathbb{N}}, x \mapsto \widehat{\sigma}_X^{-1}\{x\}$ *is strongly* $\mathbf{\Sigma}^0_2$*–computable.*

Effective Saint Raymond Section Theorem

Proposition 21 *Let X be a computable metric space and let $D := \mathrm{dom}(\delta_X)$. Then there is a computable operation $S : \mathbf{\Sigma}^0_k(D \to \mathbb{N}^\mathbb{N}) \rightrightarrows \mathbf{\Sigma}^0_2(X \to \mathbb{N}^\mathbb{N}) \times \mathbf{\Sigma}^0_k(X \to \mathbb{N}^\mathbb{N})$ for any $k \geq 2$ such that $\delta_X \circ s(x) = x$ and $g = F \circ s$ for all $\mathbf{\Sigma}^0_k$–measurable functions $F : D \to \mathbb{N}^\mathbb{N}$ and $(s, g) \in S(F)$.*



Proof. The proof can be done by induction on k. The case $k = 2$ follows from the effective Bhattacharya-Srivastava Selection Theorem. The induction step follows with the help of the Completeness Theorem. □

Effective Bhattacharya-Srivastava Selection Theorem

Definition 22 Let X, Y be computable metric spaces. A multi-valued operation $F :\subseteq X \rightrightarrows Y$ is called *strongly effectively $\mathbf{\Sigma}^0_k$–measurable* or *strongly $\mathbf{\Sigma}^0_k$–computable*, if there exists a computable operation $\Phi : \mathbf{\Pi}^0_1(Y) \rightrightarrows \mathbf{\Sigma}^0_k(X)$ such that $F^{-1}(A) = B \cap \mathrm{dom}(F)$ for any $A \in \mathbf{\Pi}^0_1(Y)$ and $B \in \Phi(A)$.

Theorem 23 *Let X, W, Z be computable metric spaces, let W be complete with recursive open balls and let $k \geq 2$. For any closed valued strongly $\mathbf{\Sigma}^0_k$–measurable $\Delta : X \rightrightarrows W$ and any given $\mathbf{\Sigma}^0_2$–measurable function $F :\subseteq W \to Z$ with $\mathrm{range}(\Delta) \subseteq \mathrm{dom}(F)$ we can effectively find a $\mathbf{\Sigma}^0_k$–measurable function $s : X \to W$ such that $s(x) \in \Delta(x)$ for any $x \in X$ and $F \circ s : X \to Z$ is $\mathbf{\Sigma}^0_k$–measurable.*

Some facts on the proof

Proof.

- The classical proof is based on a variant of a Souslin scheme.
- This construction is essentially constructive.
- Certain ineffective choices of points in the classical proof can be eliminated using multi-valued operations and the uniform limit closure scheme.
- Thus, instead of choosing certain points (which is not constructive) we can compute on all possible different points in parallel (which turns out to be constructive).

□

Reducibility of functions

Definition 24 Let X, Y, U, V be computable metric spaces and consider functions $f :\subseteq X \to Y$ and $g :\subseteq U \to V$. We say that

- f is *reducible* to g, for short $f \leqslant_{\mathrm{t}} g$, if there are continuous functions $A :\subseteq X \times V \to Y$ and $B :\subseteq X \to U$ such that
$$f(x) = A(x, g \circ B(x))$$
for all $x \in \mathrm{dom}(f)$,
- f is *computably reducible* to g, for short $f \leqslant_{\mathrm{c}} g$, if there are computable A, B as above.
- The corresponding equivalences are denoted by $\cong_{\mathrm{t}}$ and $\cong_{\mathrm{c}}$.

Proposition 25 *The following holds for all* $k \geq 1$*:*

1. $f \leqslant_{\mathrm{t}} g$ *and* g *is* $\mathbf{\Sigma}^0_k$*–measurable* $\Longrightarrow$ f *is* $\mathbf{\Sigma}^0_k$*–measurable,*
2. $f \leqslant_{\mathrm{c}} g$ *and* g *is* Σ^0_k*–computable* $\Longrightarrow$ f *is* Σ^0_k*–computable.*

Completeness Theorem for Baire space

Definition 26 For any $k \in \mathbb{N}$ we define $C_k : \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ by

$$C_k(p)(n) := \begin{cases} 0 & \text{if } (\exists n_k)(\forall n_{k-1})...\, p\langle n, n_1, ..., n_k\rangle \neq 0 \\ 1 & \text{otherwise} \end{cases}$$

for all $p \in \mathbb{N}^\mathbb{N}$ and $n \in \mathbb{N}$.

Theorem 27 *Let $k \in \mathbb{N}$. For any function $F :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ we obtain:*

1. $F \leqslant_{\mathrm{t}} C_k \iff F$ *is* $\mathbf{\Sigma}^0_{k+1}$*–measurable,*
2. $F \leqslant_{\mathrm{c}} C_k \iff F$ *is* Σ^0_{k+1}*–computable.*

Proof. Employ the Tarski-Kuratowski Normal Form in the appropriate way. □

Realizer reducibility

Definition 28 Let X, Y, U, V be computable metric spaces and consider functions $f : X \to Y$ and $g : U \to V$. We define

$$f \preceq_{\mathrm{t}} g :\iff f\delta_X \leqslant_{\mathrm{t}} g\,\delta_U$$

and we say that f is *realizer reducible* to g, if this holds. Analogously, we define $f \preceq_{\mathrm{c}} g$ with $\leqslant_{\mathrm{c}}$ instead of $\leqslant_{\mathrm{t}}$. The corresponding equivalences $\approx_{\mathrm{t}}$ and $\approx_{\mathrm{c}}$ are defined straightforwardly.

Proposition 29 *Let X, Y, U, V be computable metric spaces and consider functions $f : X \to Y$ and $g : U \to V$. Then the following holds for all $k \geq 1$:*

1. $f \preceq_{\mathrm{t}} g$ *and* g *is* $\mathbf{\Sigma}^0_k$*–measurable* $\Longrightarrow f$ *is* $\mathbf{\Sigma}^0_k$*–measurable,*
2. $f \preceq_{\mathrm{c}} g$ *and* g *is* Σ^0_k*–computable* $\Longrightarrow f$ *is* Σ^0_k*–computable.*

Characterization of realizer reducibility

Definition 30 Let X, Y, U, V be computable metric spaces, let $\mathcal{F}$ be a set of functions $F : X \to Y$ and let $\mathcal{G}$ be a set of functions $G : U \to V$. We define

$$\mathcal{F} \leqslant_{\mathrm{t}} \mathcal{G} \quad :\iff \quad (\exists A, B \text{ computable})(\forall G \in \mathcal{G})(\exists F \in \mathcal{F}) \\ (\forall x \in \mathrm{dom}(F))\, F(x) = A(x, GB(x)),$$

where $A :\subseteq X \times V \to Y$ and $B :\subseteq X \to U$. Analogously, one can define $\leqslant_{\mathrm{c}}$ with computable A, B.

Proposition 31 *Let X, Y, U, V be computable metric spaces and let $f : X \to Y$ and $g : U \to V$ be functions. Then*

$$f \preceq_{\mathrm{c}} g \iff \{F : F \vdash f\} \leqslant_{\mathrm{c}} \{G : G \vdash g\}.$$

An analogous statement holds with respect to $\preceq_{\mathrm{t}}$ and $\leqslant_{\mathrm{t}}$.

Completeness Theorem for realizer reducibility

Theorem 32 *Let X, Y be computable metric spaces and let $k \in \mathbb{N}$. For any function $f : X \to Y$ we obtain:*

1. $f \preceq_{\mathrm{t}} C_k \iff f$ *is $\mathbf{\Sigma}^0_{k+1}$–measurable,*
2. $f \preceq_{\mathrm{c}} C_k \iff f$ *is $\mathbf{\Sigma}^0_{k+1}$–computable.*

Proof. We consider the computable case (2), the topological case (1) can be proved analogously. Let f be $\mathbf{\Sigma}^0_{k+1}$–computable. Then by the Representation Theorem f admits a $\mathbf{\Sigma}^0_{k+1}$–computable realizer F and hence $F \leqslant_{\mathrm{c}} C_k$ by the Completeness Theorem. Since δ_Y is computable and $\delta_{\mathbb{N}}^{\mathbb{N}}$ admits a computable right inverse, it follows $f\delta_X = \delta_Y F \leqslant_{\mathrm{c}} C_k \delta_{\mathbb{N}}^{\mathbb{N}}$ and thus $f \preceq_{\mathrm{c}} C_k$. Now let, on the other hand, $f \preceq_{\mathrm{c}} C_k$. Since C_k is $\mathbf{\Sigma}^0_{k+1}$–computable by the Completeness Theorem, it follows that f is $\mathbf{\Sigma}^0_{k+1}$–computable. □

Definition 33 Let X, Y be computable metric spaces, let $f : X \to Y$ be a function and $k \in \mathbb{N}$. Then f is called $\mathbf{\Sigma}^0_{k+1}$*–complete*, if $f \approx_{\mathrm{c}} C_k$.

Lower bounds for unbounded closed linear operators

Theorem 34 *Let X, Y be computable Banach spaces and let $f :\subseteq X \to Y$ be a closed linear and unbounded operator. Let $(e_n)_{n\in\mathbb{N}}$ be a computable sequence in $\mathrm{dom}(f)$ whose linear span is dense in X and let $f(e_n)_{n\in\mathbb{N}}$ be computable in Y. Then $C_1 \leqslant_c f$.*

This generalizes The First Main Theorem of Pour-El and Richards.

Arithmetic complexity of points

Definition 35 Let X be a computable metric space and let $x \in X$. Then x is called Δ^0_n*–computable*, if there is a Δ^0_n–computable $p \in \mathbb{N}^\mathbb{N}$ such that $x = \delta_X(p)$.

Proposition 36 *If (X, d, α) is a computable metric space such that the equivalence problem for balls*

$$\{\langle m, k, i, j\rangle \in \mathbb{N} : B(\alpha(m), \bar{i}) = B(\alpha(k), \bar{j})\}$$

is r.e., then we obtain for any point $x \in X$ and $n \geq 1$:

$$x \text{ is } \Delta^0_n\text{–computable} \iff \{\langle m, i\rangle \in \mathbb{N} : x \in B(\alpha(m), \bar{i})\} \in \Sigma^0_n.$$

Invariance Theorem

Theorem 37 *Let X, Y be computable metric spaces.*

- *If $f : X \to Y$ is $\mathbf{\Sigma}^0_k$–computable, then it maps Δ^0_n–computable inputs $x \in X$ to Δ^0_{n+k-1}–computable outputs $f(x) \in Y$ for all $n, k \geq 1$.*
- *If f is even $\mathbf{\Sigma}^0_k$–complete and $k \geq 2$, then there is some Δ^0_n–computable input $x \in X$ for any $n \geq 1$ which is mapped to some Δ^0_{n+k-1}–computable output $f(x) \in Y$ which is not Δ^0_{n+k-2}–computable.*

Completeness of the limit

Proposition 38 *Let X be a computable metric space and let $c := \{(x_n)_{n\in\mathbb{N}} \in X^{\mathbb{N}} : (x_n)_{n\in\mathbb{N}} \in X^{\mathbb{N}} \text{ converges}\}$ denote the computable metric subspace of $X^{\mathbb{N}}$. The ordinary limit map*

$$\lim : c \to X, (x_n)_{n\in\mathbb{N}} \mapsto \lim_{n\to\infty} x_n$$

is $\mathbf{\Sigma}^0_2$–computable and it is even $\mathbf{\Sigma}^0_2$–complete, if there is a computable embedding $\iota : \{0,1\}^{\mathbb{N}} \hookrightarrow X$.

Proof. On the one hand, $\mathbf{\Sigma}^0_2$–computability follows from

$$\lim{}^{-1}(B(x,r)) = \left(\bigcup_{n=0}^{\infty} X^n \times \overline{B}(x, r - 2^{-n})^{\mathbb{N}}\right) \cap c \in \mathbf{\Sigma}^0_2(c)$$

and on the other hand, $\mathbf{\Sigma}^0_2$–completeness follows from

$$C_1 \leqslant_c \lim{}_{\{0,1\}^{\mathbb{N}}} \leqslant_c \lim{}_X .$$

□

Completeness of differentiation

Proposition 39 *Let $\mathcal{C}^{(1)}[0,1]$ be the computable metric subspace of $\mathcal{C}[0,1]$ which contains the continuously differentiable functions $f:[0,1]\to\mathbb{R}$. The operator of differentiation*

$$d:\mathcal{C}^{(1)}[0,1]\to\mathcal{C}[0,1], f\mapsto f'$$

is $\mathbf{\Sigma}_2^0$–complete.

Proof. d is a linear closed an unbounded operator which is computable on the dense sequence of rational polynomials. Hence, $C_1\leqslant_{\mathrm{c}} d$. On the other hand, we obtain

$$f'(x)=\lim_{n\to\infty}\frac{f(x+(1-x)2^{-n})-f(x-x2^{-n})}{2^{-n}}$$

for all $f\in\mathcal{C}^{(1)}[0,1]$ and $x\in[0,1]$. Thus, d can be obtained as a limit of a pointwise convergent sequence of $\mathbf{\Sigma}_1^0$–computable functions and is therefore $\mathbf{\Sigma}_2^0$–computable. □

Effective Banach-Hausdorff-Lebesgue Theorem

Theorem 40 *Let X and Y be computable metric spaces and let $k\geq 1$. There is a computable operation*

$$\Lambda:\mathbf{\Sigma}_{k+1}^0(X\to Y)\rightrightarrows\mathbf{\Sigma}_k^0(X\rightrightarrows Y^{\mathbb{N}})$$

such that $\lim\circ L=f$ for all $f\in\mathbf{\Sigma}_{k+1}^0(X\to Y)$ and $L\in\Lambda(f)$.

Corollary 41 *Let X and Y be computable metric spaces and let $k\geq 2$. Then for any $\mathbf{\Sigma}_{k+1}^0$–computable function $f:X\to Y$ there is a computable sequence $(f_n)_{n\in\mathbb{N}}$ of $\mathbf{\Sigma}_k^0$–computable functions such that $f=\lim_{n\to\infty}f_n$. For $X=\mathbb{N}^{\mathbb{N}}$ this holds true in case $k=1$ as well.*